Applications of parabolic Hecke algebras: parabolic induction and Hecke polynomials

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Abstract

The present thesis is dedicated to the study of two problems in the theory of Hecke algebras.

The first part deals with parabolic induction for modules over the pro-p Iwahori-Hecke algebra. This functor was introduced by Ollivier for the general linear group, and later on for connected reductive groups by Ollivier and Vignéras. Its properties were studied by Abe, Ollivier, and Vignéras. We redefine the parabolic induction functor by replacing the positive subalgebra with a certain parabolic Hecke algebra. To this end, we construct two algebra homomorphisms from the parabolic Hecke algebra to the Levi algebra and the pro-p Iwahori-Hecke algebra, respectively. Both homomorphisms factor over a common algebra, which can be viewed as an interpolation between the Levi algebra and the pro-p Iwahori-Hecke algebra. In order to study this new class of algebras it is necessary to investigate a certain function on the parabolic subgroup with values in the natural numbers. We establish three fundamental properties of this function: one describes its behavior under the projection map from the parabolic to the Levi subgroup, the second measures the deviation of the length functions of the Iwahori-Weyl group and its Levi counterpart, respectively, and the third property pertains to the Bruhat order on the Iwahori-Weyl group of the Levi. We define an induction functor for the new class of algebras and prove a transitivity property. As a corollary we obtain a new proof of the transitivity of parabolic induction.

The second problem concerns spherical (parahoric) Hecke algebras which has its origins in the theory of Siegel modular forms. One is interested in how polynomials with coefficients in the spherical Hecke algebra decompose over a larger parabolic Hecke algebra. Building upon the work of Andrianov and Gritsenko, we develop the theory for connected reductive groups. We conjecture that a certain Hecke polynomial has a specific "left root" in the parabolic Hecke algebra. Subsequently, we prove this statement to be true, provided the parabolic subgroup in question is contained in a *non-obtuse* parabolic. We give a classification of non-obtuse parabolics and show that many maximal parabolics share this property. Finally, we proceed to derive a general decomposition theorem.

Zusammenfassung

Die vorliegende Arbeit widmet sich dem Studium zweier Probleme in der Theorie der Heckealgebren.

Der erste Teil handelt von der parabolischen Induktion von Moduln über der pro-p Iwahori-Heckealgebra. Dieser Funktor wurde eingeführt von Ollivier für die allgemeine lineare Gruppe, und später für zusammenhängende reduktive Gruppen von Ollivier und Vignéras. Seine Eigenschaften wurden von Abe, Ollivier und Vignéras untersucht. Wir geben eine neue Definition des parabolischen Induktionsfunktors, wobei wir die positive Unteralgebra durch eine gewisse parabolische Heckealgebra ersetzen. Zu diesem Zweck konstruieren wir zwei Homomorphismen von der parabolischen Heckealgebra auf jeweils die Levialgebra und die pro-p Iwahori-Heckealgebra. Beide Homomorphismen faktorisieren über eine gemeinsame Algebra, welche als eine Interpolation zwischen der Levialgebra und der pro-p Iwahori-Heckealgebra angesehen werden kann. Um diese neue Klasse von Algebren zu verstehen, ist es notwendig, eine gewisse Funktion auf der parabolischen Untergruppe mit Werten in den natürlichen Zahlen zu untersuchen. Wir weisen drei fundamentale Eigenschaften dieser Funktion nach: Die erste beschreibt ihr Verhalten unter der Projektionsabbildung von der Parabolischen auf die Leviuntergruppe; die zweite misst den Unterschied zwischen den Längenfunktionen der Iwahori-Weylgruppe und ihrem Pendant für die Leviuntergruppe; die dritte Eigenschaft betrifft die Bruhatordnung der Iwahori-Weylgruppe der Leviuntergruppe. Wir definieren einen Induktionsfunktor für die neue Klasse von Algebren und beweisen eine Transitivitätseigenschaft. Als Korollar erhalten wir einen neuen Beweis der Transitivität der parabolischen Induktion.

Das zweite Problem, welches seinen Ursprung in der Theorie der Siegelschen Modulformen hat, befasst sich mit sphärischen (parahorischen) Heckealgebren. Man interessiert sich dafür, wie Polynome mit Koeffizienten in der sphärischen Heckealgebra über der größeren parabolischen Heckealgebra zerfallen. Aufbauend auf der Arbeit von Andrianov und Gritsenko entwickeln wir eine Theorie für zusammenhängende reduktive Gruppen. Wir vermuten, dass ein gewisses Heckepolynom eine spezielle „Linksnullstelle“ in der parabolischen Heckealgebra besitzt. Anschließend beweisen wir diese Vermutung unter der Voraussetzung, dass die zugrunde liegende parabolische Untergruppe in einer *nichtstumpfen* Parabolischen enthalten ist. Wir geben eine Klassifikation der nichtstumpfen Parabolischen an und zeigen, dass viele maximale Parabolische diese Eigenschaft besitzen. Schließlich folgern wir einen allgemeinen Zerlegungssatz.

Contents

0. Introduction

0.1. Parabolic induction

A driving force of research in number theory in the past 25 years has been the mod-p local Langlands program. Given a non-archimedean local field F of residue characteristic $p > 0$, one tries to match, on the one hand, certain continuous n-dimensional $\overline{\mathbb{F}}_p$-representations of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F and, on the other hand, certain smooth admissible irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(F)$. It is therefore of interest to study the smooth $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(F)$.

A first step in this direction was made by Barthel and Livné in 1994 [BL94] by giving a "rough" classification of smooth irreducible representations of $\mathrm{GL}_2(F)$ on $\overline{\mathbb{F}}_p$-vector spaces. The smooth representation theory of $\mathrm{GL}_n(F)$ on complex vector spaces had been well-understood (see e. g. [BZ76]), and Vignéras subsequently adapted the theory to study the smooth irreducible $\mathrm{GL}_2(F)$-representations over $\overline{\mathbb{F}}_\ell$ for $\ell \neq p$ [Vig89]. In the complex setting the category $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ of smooth representations of $\mathrm{GL}_n(F)$ is equivalent to the category of modules over the (global) Hecke algebra of $\mathrm{GL}_n(F)$, i. e. the algebra of locally constant functions $\mathrm{GL}_n(F) \to \mathbb{C}$ with compact support, where multiplication is defined via integration with respect to a Haar measure. Here, the importance of the Hecke algebra is self-evident.

However, many of the methods in the classical case fail spectacularly for representations over $\overline{\mathbb{F}}_p$: for instance there are no non-zero $\overline{\mathbb{F}}_p$-valued Haar measures on $\mathrm{GL}_n(F)$, since p divides the pro-order of any compact open subgroup of $\mathrm{GL}_n(F)$. But then the usual Hecke algebra does not exist. What is more, the lack of non-trivial smooth characters of F scotches the development of a Fourier theory or Whittaker models.

Nevertheless, there is a silver-lining. It is due to the elementary fact that any p-group acting on a finite set with cardinality not divisible by p leaves an element fixed. It follows readily that any pro-p group acting smoothly on an $\overline{\mathbb{F}}_p$-vector space has a non-zero invariant vector. Denote by $I(1)$ the standard pro-p Iwahori subgroup of $\mathrm{GL}_n(F)$ and call the endomorphism ring

$$\mathcal{H}_R := \mathrm{End}_{\mathrm{GL}_n(F)}\big(\mathrm{ind}_{I(1)}^{\mathrm{GL}_n(F)}(\mathbb{1})\big)$$

the pro-p Iwahori-Hecke algebra, where $\mathrm{ind}_{I(1)}^{\mathrm{GL}_n(F)}(\mathbb{1})$ denotes the compact induction from the trivial R-representation of $I(1)$ to $\mathrm{GL}_n(F)$ in the category of smooth $\mathrm{GL}_n(F)$-representations on R-modules, and where R is a commutative unital ring. Thus, taking $I(1)$-invariants provides a functor

$$\begin{aligned} \mathrm{Rep}_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(F)) &\longrightarrow \mathrm{Mod}(\mathcal{H}_{\overline{\mathbb{F}}_p}), \\ V &\longmapsto \mathrm{Hom}_{\mathrm{GL}_n(F)}\big(\mathrm{ind}_{I(1)}^{\mathrm{GL}_n(F)}(\mathbb{1}), V\big) \cong V^{I(1)} \end{aligned} \tag{0.1.1}$$

from the category of smooth $\mathrm{GL}_n(F)$-representations to the category of right $\mathcal{H}_{\overline{\mathbb{F}}_p}$-modules. This functor is not exact, though, but it has a left adjoint given by

$$\mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{H}_{\overline{\mathbb{F}}_p}} \mathrm{ind}_{I(1)}^{\mathrm{GL}_n(F)}(\mathbb{1}).$$

Thus, one can hope for a close connection between right $\mathcal{H}_{\overline{\mathbb{F}}_p}$-modules and those smooth $\mathrm{GL}_n(F)$-representations which are generated by their $I(1)$-invariant vectors. In her article [Vig04] Vignéras exploited this connection, after which it became apparent that the pro-p Iwahori-Hecke algebra played a fundamental role in understanding the smooth representation theory of $\mathrm{GL}_n(F)$.

Subsequently, Vignéras set out to study the pro-p Iwahori-Hecke algebra systematically starting with [Vig05], replacing $\mathrm{GL}_n(F)$ for the sake of generality by a connected split reductive group over F. The remarkable insights in Schmidt's Diplomarbeit [Sch09] finally led to the more definitive treatment [Vig16], which gives a structure theory for the pro-p Iwahori-Hecke algebra of a general connected reductive group and realizes it as a generic pro-p Hecke algebra.

In order to study right $\mathcal{H}_{\overline{\mathbb{F}}_p}$-modules it is natural to try to imitate the methods used to study smooth $\mathrm{GL}_n(F)$-representations. Let us recall the parabolic induction functor: let P be a parabolic subgroup of $\mathrm{GL}_n(F)$, say the subgroup of block-upper triangular matrices. Its Levi subgroup M is then isomorphic to $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$ with $n_1 + \cdots + n_r = n$. We have a semidirect product $P = M \ltimes U_P$, where U_P is the unipotent radical of P. Given a smooth M-representation V, we view it as a smooth P-representation via inflation along the canonical quotient map $P \twoheadrightarrow M$. The smooth induction $\mathrm{Ind}_P^{\mathrm{GL}_n(F)}(V)$ then provides a smooth $\mathrm{GL}_n(F)$-representation in a functorial way. For varying proper parabolics these functors prove to be very important for the classification of the irreducible representations of $\mathrm{GL}_n(F)$, see [Her11b].

It is not obvious, though, how to translate this functor into the world of Hecke modules. Bushnell and Kutzko [BK98] described the smooth complex representations of a reductive p-adic group via the theory of types. Consider the pro-p Iwahori-Hecke algebra $\mathcal{H}_R(M) = \mathrm{End}_M(\mathrm{ind}_{I_M(1)}^M(\mathbb{1}))$ of M, where $I_M(1) := I(1) \cap M$, for any coefficient ring R. It is the localization at a central element of the positive subalgebra $\mathcal{H}_R(M^+)$ of $\mathcal{H}_R(M)$, which is attached to a certain submonoid $M^+ \subseteq M$ of positive elements. Bushnell and Kutzko define a canonical embedding $\mathcal{H}_{\mathbb{C}}(M^+) \hookrightarrow \mathcal{H}_{\mathbb{C}}$, which extends uniquely to an embedding $\mathcal{H}_{\mathbb{C}}(M) \hookrightarrow \mathcal{H}_{\mathbb{C}}$. It provides a functor

$$\mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{H}_{\mathbb{C}}(M)} \mathcal{H}_{\mathbb{C}}$$

from the category of complex right $\mathcal{H}_{\mathbb{C}}(M)$-modules to the category of complex right $\mathcal{H}_{\mathbb{C}}$-modules. This embedding, however, does not exist when we consider Hecke algebras over $\overline{\mathbb{F}}_p$. Still, the positive subalgebra $\mathcal{H}_R(M^+)$ *always* embeds into $\mathcal{H}_R$ for any R, see [BK98] or [Vig98]. Realizing this, Ollivier defined an induction functor via

$$\mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)} \mathcal{H}_{\overline{\mathbb{F}}_p},$$

which works even in characteristic p, and studied its properties in [Oll10]. This turned out to be the correct notion of parabolic induction, which is also reflected in the amount of attention it received, see [OV18, Vig15, Abe16a, Abe16b]. In particular, it is compatible with parabolic induction for smooth $\mathrm{GL}_n(F)$-representations.

However, despite being justified by its success this definition of parabolic induction defies intuition. Indeed, it only gives *some* way of functorially associating a right $\mathcal{H}_{\overline{\mathbb{F}}_p}$-

module with a right $\mathcal{H}_{\overline{\mathbb{F}}_p}(M)$-module, while the role of P is not clear. By comparison: what happens if we bypass the parabolic subgroup and consider the functor $V \mapsto \mathrm{Ind}_M^{\mathrm{GL}_n(F)}(V)$ from $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(M)$ to $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(F))$? The answer is that we obtain representations that are too complicated to study. Here it is necessary to inflate before inducing. One would therefore also expect P to play an influential role in the parabolic induction for pro-p Iwahori-Hecke modules.

Moreover, the positive subalgebra $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$ itself remains mysterious. It became clear only in hindsight that $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$ is "at the core" of (the correct notion of) parabolic induction. But restricting consideration to this algebra also has its drawback. Since M^+ is only a monoid $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$ is not the endomorphism ring of a compactly induced representation. One reason why this might potentially be a problem is grounded in the following issue: in [Sch15] Schneider defines a certain differential graded algebra $\mathcal{H}^\bullet$, called Hecke DGA, whose 0-th cohomology coincides with $\mathcal{H}_{\overline{\mathbb{F}}_p}$. When $I(1)$ is torsion-free, the main result of [Sch15] implies that the functor (0.1.1) induces an equivalence of triangulated categories

$$D\big(\mathrm{GL}_n(F)\big) \xrightarrow{\sim} D(\mathcal{H}^\bullet),$$

where $D(\mathrm{GL}_n(F))$ is the full derived category of $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(\mathrm{GL}_n(F))$ and $D(\mathcal{H}^\bullet)$ is the derived category of differential graded modules over $\mathcal{H}^\bullet$. In fact, it shows that the equivalence even holds when we replace $\mathrm{GL}_n(F)$ with (the F-points of) an arbitrary connected reductive group $\mathbf{G}$. This surprising result suggests that the derived setting is the correct framework to study the connection between smooth $\mathrm{GL}_n(F)$-representations and $\mathcal{H}_{\overline{\mathbb{F}}_p}$-modules. In studying $\mathcal{H}^\bullet$ it becomes natural to ask for a parabolic induction functor for Hecke DGA's. While it is easy to define a derived version of a Hecke algebra, it is not clear how a derived version of $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$ should look like. Therefore, it is desirable to replace $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$ by a genuine Hecke algebra.

We are therefore led to ask the following question: can we define parabolic induction by replacing $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$ with a "parabolic" Hecke algebra? One purpose of this thesis is to answer this question.

To this end, we make the following definition:

Definition 0.1. We call $\mathcal{H}_R(P) := \mathrm{End}_P(\mathrm{ind}_{I_P(1)}^P(\mathbb{1}))$ the *parabolic Hecke algebra*, where $I_P(1) := I(1) \cap P$ and R is a commutative unital ring.

Our goal, now, is to find two algebra homomorphisms

$$\begin{array}{ccc} & \mathcal{H}_{\overline{\mathbb{F}}_p}(P) & \\ \swarrow & & \searrow \\ \mathcal{H}_{\overline{\mathbb{F}}_p} & & \mathcal{H}_{\overline{\mathbb{F}}_p}(M). \end{array}$$

Given these maps, it is straightforward to define a functor

$$\mathrm{Mod}(\mathcal{H}_{\overline{\mathbb{F}}_p}(M)) \to \mathrm{Mod}(\mathcal{H}_{\overline{\mathbb{F}}_p})$$

by tensoring a right $\mathcal{H}_{\overline{\mathbb{F}}_p}(M)$-module with $\mathcal{H}_{\overline{\mathbb{F}}_p}(M) \otimes_{\mathcal{H}_{\overline{\mathbb{F}}_p}(P)} \mathcal{H}_{\overline{\mathbb{F}}_p}$.

We replace $\overline{\mathbb{F}}_p$ with an arbitrary commutative unital ring R. It turns out to be advantageous to work with *abstract Hecke rings*, as developed in [AZ95], rather than with endomorphism rings. Then $\mathcal{H}_R(P)$ is a free R-module with basis

$$\left\{ \left(I_P(1) g I_P(1)\right) \,\middle|\, I_P(1) g I_P(1) \in I_P(1)\backslash P / I_P(1) \right\}.$$

We start by constructing a homomorphism $\Theta^P_{M,R} \colon \mathcal{H}_R(P) \to \mathcal{H}_R(M)$. The projection map $P \twoheadrightarrow M$, $g \mapsto g_M$ induces a linear map on the invariant spaces $\left(\mathrm{ind}^P_{I_P(1)}(\mathbb{1})\right)^{I_P(1)} \to \left(\mathrm{ind}^M_{I_M(1)}(\mathbb{1})\right)^{I_M(1)}$. Now, by Frobenius reciprocity this is equivalent to giving a map $\Theta^P_{M,R} \colon \mathcal{H}_R(P) \to \mathcal{H}_R(M)$. We obtain the following concrete description of $\Theta^P_{M,R}$:

Proposition 0.2. *The map $\Theta^P_{M,R} \colon \mathcal{H}_R(P) \to \mathcal{H}_R(M)$ is a morphism of R-algebras. Viewing $\mathcal{H}_R(P)$ and $\mathcal{H}_R(M)$ as abstract Hecke algebras, it is given by*

$$\left(I_P(1) g I_P(1)\right) \longmapsto \nu_M(g)\mu_{U_P}(g) \cdot \left(I_M(1) g_M I_M(1)\right).$$

Here, $\left\{ (I_M(1) g I_M(1)) \,\middle|\, I_M(1) g I_M(1) \in I_M(1)\backslash M / I_M(1) \right\}$ is the distinguished R-basis of $\mathcal{H}_R(M)$ as an abstract Hecke algebra. The coefficients $\nu_M(g)$ and $\mu_{U_P}(g)$ are certain p-powers depending only on the double coset of g with respect to $I_P(1)$. The numbers $\nu_M(g)$ appeared unexpectedly while studying the decomposition of $I_P(1) g I_P(1)$ into right cosets with regard to M and U_P. For the purposes of this introduction it is sufficient to know that $\nu_M(g) = 1$ whenever $g \in M$. The numbers $\mu_{U_P}(g)$, however, play a major role in our subsequent observations. They are defined as the indices

$$\mu_{U_P}(g) := [I_{U_P} : I_{U_P} \cap g^{-1} I_{U_P} g],$$

where $I_{U_P} := I(1) \cap U_P$. We observe that the map $P \to p^{\mathbb{Z}}$, $g \mapsto \mu_{U_P}(g)/\mu_{U_P}(g^{-1})$ is actually the modulus character δ_P of P. While δ_P is ubiquitous in the theory of smooth complex representations, it is useless in the context of mod-p representations. One might therefore expect $\mu_{U_P} \colon P \to p^{\mathbb{Z}_{\geq 0}}$ to be an appropriate substitute for δ_P.

In order to understand the image of $\Theta^P_{M,R}$, we characterize the monoid M^+, which is used to define the positive subalgebra $\mathcal{H}_R(M^+)$, as the set of all $m \in M$ with $m I_{U_P} m^{-1} \subseteq I_{U_P}$ or, equivalently, $\mu_{U_P}(m) = 1$. This means that elements $m \in M$ with $\mu_{U_P}(m) = 1$ automatically satisfy $I_{U_{P^{\mathrm{op}}}} \subseteq m I_{U_{P^{\mathrm{op}}}} m^{-1}$, where $U_{P^{\mathrm{op}}}$ is the unipotent radical of the parabolic opposite to P with Levi subgroup M. The characterization is crucial, for it tells us that M^+ is entirely determined by I_{U_P}. This observation explains the subtle role P plays in Ollivier's parabolic induction functor [Oll10]. Since we are working with matrices, it is easy to prove that if $g \in P$ satisfies $\mu_{U_P}(g) = 1$, then its image g_M lies in M^+. We therefore deduce

Corollary 0.3. *The image of $\Theta^P_{M,R}$ contains $\mathcal{H}_R(M^+)$. If $R = \overline{\mathbb{F}}_p$, then the image of $\Theta^P_{M,\overline{\mathbb{F}}_p}$ coincides with $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$.*

In the case $R = \overline{\mathbb{F}}_p$ one is thus tempted to define a homomorphism of $\overline{\mathbb{F}}_p$-algebras $\widetilde{\Theta}^P_{\mathrm{GL}_n(F),\overline{\mathbb{F}}_p} : \mathcal{H}_{\overline{\mathbb{F}}_p}(P) \to \mathcal{H}_{\overline{\mathbb{F}}_p}$ as the composite of $\Theta^P_{M,\overline{\mathbb{F}}_p}$ with the inclusion homomorphism $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+) \hookrightarrow \mathcal{H}_{\overline{\mathbb{F}}_p}$. In a superficial way this achieves our objective of defining parabolic induction using $\mathcal{H}_{\overline{\mathbb{F}}_p}(P)$. But this solution is not satisfactory, for it works only in characteristic p. Hence, we ask more ambitiously whether we can remove the restriction on the coefficient ring.

We are even more ambitious and, following the predominant sentiment, replace $\mathrm{GL}_n(F)$ with $G := \mathbf{G}(F)$ for a connected (possibly non-split) reductive group $\mathbf{G}$ defined over F. In general, we write an algebraic group in a boldface letter, whereas its group of F-points is denoted by the same non-boldface letter. Let $\mathbf{T}$ be a maximal F-split torus in $\mathbf{G}$. We fix a minimal parabolic $\mathbf{B}$ containing $\mathbf{T}$ and consider the pro-p Iwahori subgroup $I(1)$ associated with $\mathbf{B}$. We consider the pro-p Iwahori-Hecke algebra

$$\mathcal{H}_R := \mathrm{End}_G\big(\mathrm{ind}^G_{I(1)}(\mathbb{1})\big).$$

Let $\mathbf{P}$ be a (standard) parabolic with Levi decomposition $\mathbf{P} = \mathbf{M}\mathbf{U}_P$. Most results that we proved for $\mathrm{GL}_n(F)$ straightforwardly carry over to this more general setting. In particular, we have the same characterization of M^+ and a homomorphism $\Theta^P_{M,R} : \mathcal{H}_R(P) \to \mathcal{H}_R(M)$. The image of $\Theta^P_{M,R}$ again contains $\mathcal{H}_R(M^+)$. But precisely determining the image is a major issue; even in the case $R = \overline{\mathbb{F}}_p$ it is not obvious that it coincides with $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$. To this end, we need to achieve a better understanding of the $\mu_{U_P}(g)$. The key result we obtain is the following inequality:

Proposition 0.4. *We have $\mu_{U_P}(g_M) \le \mu_{U_P}(g)$ for all $g \in P$.*

This inequality is not surprising. It quickly becomes clear from examples that the intuitive reason for this result is that conjugating $g_M^{-1} I_{U_P} g_M$ by $g_{U_P}^{-1}$, where $g_{U_P} := g_M^{-1} g \in U_P$ is the unipotent part of $g \in P$, results in a "twisting", in the sense that single root groups give contributions to higher root groups; hence it is plausible that $I_{U_P} \cap g^{-1} I_{U_P} g$ is in some sense smaller than $I_{U_P} \cap g_M^{-1} I_{U_P} g_M$, even if there is no containment relation between these groups. The appropriate framework to study this phenomenon is the one of groups with a root group datum, which is a part of Bruhat-Tits theory. In this direction we obtain a general description of certain endomorphisms of groups that are generated by root groups corresponding to positive roots (Proposition 1.9). We then use this analysis to control the "twisting" and prove the inequality. This is one of the most technical parts of this thesis.

As an application of Proposition 0.4 we can describe the image of $\Theta^P_{M,R}$.

Corollary 0.5. *The image of $\Theta^P_{M,R}$ is generated by $\big\{\mu_{U_P}(m)(I_M(1) m I_M(1)) \,\big|\, m \in M\big\}$ as an R-module. In particular, if $R = \overline{\mathbb{F}}_p$, the image coincides with $\mathcal{H}_{\overline{\mathbb{F}}_p}(M^+)$.*

This corollary paves the way for constructing a morphism $\Xi^P_{G,R} : \mathcal{H}_R(P) \to \mathcal{H}_R$. One would like to extend the embedding $\mathcal{H}_R(M^+) \hookrightarrow \mathcal{H}_R$ to the image of $\Theta^P_{M,R}$, but, as already mentioned, this is not always possible. However, it *is* possible if $R = \mathbb{Z}$. Actually, even more is true:

Proposition 0.6. $\operatorname{Im}\Theta^P_{M,\mathbb{Z}}$ *is the maximal subalgebra of* $\mathcal{H}_\mathbb{Z}(M)$ *to which the embedding* $\mathcal{H}_\mathbb{Z}(M^+) \hookrightarrow \mathcal{H}_\mathbb{Z}$ *extends uniquely. In particular, precomposing the embedding* $\operatorname{Im}\Theta^P_{M,\mathbb{Z}} \hookrightarrow \mathcal{H}_\mathbb{Z}$ *with* $\Theta^P_{M,\mathbb{Z}}$ *provides a homomorphism* $\Xi^P_{G,\mathbb{Z}}\colon \mathcal{H}_\mathbb{Z}(P) \to \mathcal{H}_\mathbb{Z}$ *of rings. We obtain an* R*-algebra homomorphism* $\Xi^P_{G,R}\colon \mathcal{H}_R(P) \to \mathcal{H}_R$ *for any commutative unital ring* R *via extension of scalars.*

Let us give an outline of the construction. In order to do so, we need to talk about certain aspects of the structure theory of $\mathcal{H}_\mathbb{Z}$ as developed by Vignéras in [Vig16]. Let $W(1)$ be the pro-p Iwahori-Weyl group of G; it contains the pro-p Iwahori-Weyl group $W_M(1)$ of M. Denote by $(T_w)_{w\in W(1)}$ the Iwahori-Matsumoto basis of $\mathcal{H}_\mathbb{Z}$. The numbers $q_w := |I(1)\backslash I(1)wI(1)|$, for $w \in W(1)$,[1] play an important role in the structure theory of $\mathcal{H}_\mathbb{Z}$. They are powers of the cardinality q of the residue field κ_F of F. For example, if $\mathbf{G}$ is F-split, then we have $q_w = q^{\ell(w)}$, where ℓ is the length function on $W(1)$. When $\mathbf{G}$ is not F-split, this relation is not true anymore. However, these numbers still serve as a substitute for the length function, because we always have the equivalence $q_{vw} = q_v q_w \iff \ell(vw) = \ell(v) + \ell(w)$. Moreover, one defines elements $T^*_w \in T_w + \sum_{v<w}\mathbb{Z}.T_v$ in $\mathcal{H}_\mathbb{Z}$ such that $T_w T^*_{w^{-1}} = q_w$ for all $w \in W(1)$. (Here, "<" denotes the Bruhat-order in $W(1)$.) Thus, in some sense, the q_w give a measure of invertibility of the T_w.

One ingredient in proving the first statement in Proposition 0.6 is the following generalization of the Fundamental Lemma [Vig05, Lem. 13]:

$$q_{v,w}T_v^{-1}T_{vw} \in T_w + \sum_{w'<w}\mathbb{Z}.T_{w'} \subseteq \mathcal{H}_\mathbb{Z}, \quad \text{for } v,w \in W(1), \tag{0.1.2}$$

where $q_{v,w} = (q_v q_w q_{vw}^{-1})^{1/2} \in \mathbb{N}$, and where the computation takes place in $\mathcal{H}_{\mathbb{Z}[p^{-1}]}$.

The second ingredient is a description of how the embedding $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(M^+) \hookrightarrow \mathcal{H}_{\mathbb{Z}[p^{-1}]}$ extends to the full algebra $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(M)$. Let $\lambda \in W_M(1)$ be a strictly positive element. Denote by $(T^M_w)_{w\in W_M(1)}$ the Iwahori-Matsumoto basis of $\mathcal{H}_\mathbb{Z}(M)$. Then $\mathcal{H}_\mathbb{Z}(M)$ is the localization of $\mathcal{H}_\mathbb{Z}(M^+)$ at the central element T^M_λ. Put differently, T^M_λ is invertible in $\mathcal{H}_\mathbb{Z}(M)$, and for every $w \in W_M(1)$ we have $(T^M_\lambda)^n \cdot T^M_w = T^M_{\lambda^n w} \in \mathcal{H}_\mathbb{Z}(M^+)$ for $n \gg 0$. The embedding $\theta^+\colon \mathcal{H}_\mathbb{Z}(M^+) \hookrightarrow \mathcal{H}_\mathbb{Z}$ is the restriction of the linear map

$$\mathcal{H}_\mathbb{Z}(M) \longrightarrow \mathcal{H}_\mathbb{Z}, \quad T^M_w \longmapsto T_w.$$

Since T_λ is invertible in $\mathcal{H}_{\mathbb{Z}[p^{-1}]}$ the universal property of localization gives the extension

$$\mathcal{H}_{\mathbb{Z}[p^{-1}]}(M) \longrightarrow \mathcal{H}_{\mathbb{Z}[p^{-1}]}, \quad T^M_w \longmapsto T^{-1}_{\lambda^n}\cdot T_{\lambda^n w} \qquad (n \gg 0).$$

Using (0.1.2) we observe that $q_{\lambda^n,w}\cdot T^{-1}_{\lambda^n}\cdot T_{\lambda^n w} \in \mathcal{H}_\mathbb{Z}$. This means that θ^+ extends to the free subgroup A of $\mathcal{H}_\mathbb{Z}(M)$ generated by elements of the form $q_{\lambda^n,w}T^M_w$, for $w \in W_M(1)$ and $n \gg 0$.

[1]The double coset $I(1)wI(1)$ is well-defined as it does not depend on the choice of lift of w.

The third, and last, ingredient is to prove that A coincides with $\operatorname{Im}\Theta^P_{M,\mathbb{Z}}$. This follows from the observation that the function $\mu_{U_P}: M \to \mathbb{N}$ induces in a canonical way a map $\mu_{U_P}: W_M(1) \to \mathbb{N}$, and from the fact that

$$q_{\lambda^n,w} = \mu_{U_P}(w), \tag{0.1.3}$$

for any $w \in W_M(1)$ and $n \gg 0$. In particular, $q_{\lambda^n,w}$ is eventually independent of n.

This completes the construction of $\Xi^P_{G,R}$, hence also of the parabolic induction functor.

Theorem 0.7. *The functor* $\mathrm{Mod}(\mathcal{H}_R(M)) \to \mathrm{Mod}(\mathcal{H}_R)$ *given by*

$$\mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{H}_R(M)} \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(P)} \mathcal{H}_R \cong \mathcal{M} \otimes_{\mathcal{H}_R(P)} \mathcal{H}_R,$$

coincides with the parabolic induction of Ollivier and Vignéras [OV18].

Considering the function $\mu_{U_P}: W_M(1) \to \mathbb{N}$, the natural question arises whether it contains information about the structure of $W_M(1)$. Notice that $W_M(1)$ is naturally equipped with a multiplication, the length function ℓ_M, and the Bruhat order $\leq_M$. As above we observe that the map $W_M(1) \to p^{\mathbb{Z}}$, $w \mapsto \mu_{U_P}(w)/\mu_{U_P}(w^{-1})$ is a group homomorphism. Most surprising is that μ_{U_P} encodes information pertaining to ℓ_M and $\leq_M$.

Recall that we may replace ℓ_M by the map $w \mapsto q_{M,w} := |I_M(1)\backslash I_M(1)wI_M(1)|$ from $W_M(1)$ to $\mathbb{N}$. We then prove the formula

$$q_w = \mu_{U_P}(w)\mu_{U_P}(w^{-1}) \cdot q_{M,w}, \qquad \text{for all } w \in W_M(1). \tag{0.1.4}$$

Therefore, the map μ_{U_P} gives us the means to measure the deviation between $\ell|_{W_M(1)}$ and ℓ_M. We also have $\mu_{U_{P^{\mathrm{op}}}}(w) = \mu_{U_P}(w^{-1})$ for $w \in W_M(1)$, allowing for a more balanced formulation of (0.1.4).[2]

Finally, $\mu_{U_P}: W_M(1) \to \mathbb{N}$ is order preserving, i. e. we have

$$v \leq_M w \implies \mu_{U_P}(v) \leq \mu_{U_P}(w). \tag{0.1.5}$$

To M^+ is attached a certain submonoid $W_{M^+}(1)$ of $W_M(1)$, which we characterize as the set of those elements w of $W_M(1)$ with $\mu_{U_P}(w) = 1$. We then recover an easy proof of the known fact that $W_{M^+}(1)$ is a *lower subset* of $W_M(1)$, i. e. given $w \in W_{M^+}(1)$ and $v \in W_M(1)$ with $v \leq_M w$, we have $v \in W_{M^+}(1)$. Compare this with the original proof, which is spread over [Oll15, Lem. 2.9.ii], [Oll14, Fact ii], and [Abe16a, Lem. 4.1].

Having obtained a new definition of parabolic induction, we were asking whether the same methodes could be applied to define parabolic induction for Hecke DGA's. We propose a candidate for an analogue of Θ^P_M. However, it is not clear that this is the correct one, leading us to let this topic rest for the time being. Our thoughts on this issue have been assembled in an appendix.

[2] Again, P^{op} denotes the parabolic opposite to P with Levi M, and $U_{P^{\mathrm{op}}}$ denotes its unipotent radical.

Instead, our investigations went in another direction. Namely, we observe that the homomorphisms Θ_M^P and Ξ_G^P both factor through the R-algebra

$$\mathcal{H}_R(M,G) := R \otimes_{\mathbb{Z}} \operatorname{Im} \Theta_{M,\mathbb{Z}}^P, \tag{0.1.6}$$

and that replacing $\mathcal{H}_R(P)$ by $\mathcal{H}_R(M,G)$ yields the same parabolic induction functor. Doing so might seem dubious at first, since $\mathcal{H}_R(M,G)$ is in general not a Hecke algebra. But the class of algebras, obtained from (0.1.6) by letting (M,G) vary, contains the class of pro-p Iwahori-Hecke algebras, since we have $\mathcal{H}_R(G,G) = \mathcal{H}_R$. As the notation suggests, $\mathcal{H}_R(M,G)$ can be seen as an interpolation between $\mathcal{H}_R(M)$ and $\mathcal{H}_R$. Given Levi subgroups $\mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{G}$, we define homomorphisms

$$\begin{aligned} \theta_M^{L,G} &: \mathcal{H}_R(M,G) \longrightarrow \mathcal{H}_R(M,L) \quad \text{and} \\ \xi_{L,M}^G &: \mathcal{H}_R(M,G) \longrightarrow \mathcal{H}_R(L,G), \end{aligned}$$

which satisfy certain compatibility conditions. It turns out that $\theta_M^{L,G}$ is a localization homomorphism and $\xi_{L,M}^G$ is an embedding. For this class of algebras we are naturally given an induction functor: if $\mathbf{M}$, $\mathbf{L}$, and $\mathbf{M}'$ are Levi subgroups in $\mathbf{G}$ such that

$$\begin{array}{ccc} \mathbf{L} & \subseteq & \mathbf{G} \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathbf{M} & \subseteq & \mathbf{M}', \end{array}$$

then we have a functor $\operatorname{Mod}(\mathcal{H}_R(M,L)) \to \operatorname{Mod}(\mathcal{H}_R(M',G))$ given by

$$\mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{H}_R(M,L)} \mathcal{H}_R(M,L) \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(M',G).$$

Our main result on the algebras $\mathcal{H}_R(M,G)$ is the following transitivity property:

Theorem 0.8. *Let* $\mathbf{M}$, $\mathbf{M}'$, $\mathbf{M}''$, $\mathbf{L}$, *and* $\mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$ *such that*

$$\begin{array}{ccccc} \mathbf{L} & \subseteq & \mathbf{L}' & \subseteq & \mathbf{G} \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\ \mathbf{M} & \subseteq & \mathbf{M}' & \subseteq & \mathbf{M}''. \end{array}$$

Then the map

$$\begin{gathered} \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M'',G) \longrightarrow \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M',L') \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G), \\ x \otimes y \longmapsto x \otimes 1 \otimes y \end{gathered}$$

is an isomorphism of $\mathcal{H}_R(M,L)$-$\mathcal{H}_R(M'',G)$*-bimodules.*

As a corollary, we obtain another proof of the transitivity of parabolic induction, which is more natural and concrete than the one given in [Vig15, Prop. 4.3].

0.2. Decomposition of Hecke polynomials

The second topic of this thesis deals with a simple criterion for a polynomial with coefficients in a spherical Hecke algebra to decompose over a larger parabolic Hecke algebra.

The problem to decompose Hecke polynomials emerged in the theory of Hecke operators on spaces of Siegel modular forms [And77], where one of the principal tasks is to find and study the relations between Fourier coefficients of eigenfunctions of Hecke operators and the corresponding eigenvalues. Let us give a simple example. Consider the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Recall, that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is called a *modular form of weight k* if it satisfies

$$f(z) = (f|\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \mathbb{H},$$

and if it has a Fourier expansion of the form

$$f(z) = \sum_{j=0}^{\infty} \alpha_f(j) \cdot e^{2\pi i j z}.$$

We denote by $\mathfrak{M}_k$ the $\mathbb{C}$-vector space of modular forms of weight k. Let S be the set of 2×2-matrices with integral entries and positive determinant. Then the algebra of Hecke operators $\mathcal{H} := H_{\mathbb{C}}(\Gamma, S)$ naturally acts on $\mathfrak{M}_k$. We denote this action by $(f, T) \mapsto f|T$ for $f \in \mathfrak{M}_k$, $T \in \mathcal{H}$. Since the algebra $\mathcal{H}$ is commutative and $\mathfrak{M}_k$ is finite-dimensional, it follows that $\mathfrak{M}_k$ contains a basis of simultaneous eigenvectors, also called eigenfunctions, for $\mathcal{H}$. If f is an eigenfunction, we write $\lambda_f : \mathcal{H} \to \mathbb{C}$ for the corresponding eigenvalue, so that we have $f|T = \lambda_f(T) \cdot f$ for all $T \in \mathcal{H}$. Then f is an eigenfunction if and only if $\alpha_{f|T}(j) = \lambda_f(T) \cdot \alpha_f(j)$ for all $T \in \mathcal{H}$, $j \in \mathbb{N}_0$.

It is of interest to find relations between the Fourier coefficients $\alpha_f(j)$ of the eigenfunction f and the corresponding eigenvalues $\lambda_f(T)$. In order to do this let us fix a prime number p and consider the Hecke polynomial

$$Q_p(t) = 1 - (\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma) \cdot t + p \cdot (\Gamma \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \Gamma) \cdot t^2 \in \mathcal{H}[t].$$

It decomposes over the parabolic Hecke algebra $H_{\mathbb{C}}(\Gamma_0, S_0)$, where Γ_0 (resp. S_0) is the subgroup of upper triangular matrices in Γ (resp. S), into

$$Q_p(t) = \big(1 - (\Gamma_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0) \cdot t\big) \cdot \big(1 - (\Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0) \cdot t\big). \tag{0.2.1}$$

Given an eigenfunction $f \in \mathfrak{M}_k$, we consider the complex polynomial

$$Q_{p,f}(t) := 1 - \lambda_f(\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma) \cdot t + p \cdot \lambda_f(\Gamma \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \Gamma) \cdot t^2 \in \mathbb{C}[t].$$

One can use the decomposition (0.2.1) to prove that

$$Q_{p,f}(t) \cdot \sum_{j=0}^{\infty} p^{j(1-k)} \alpha_f(p^j a) t^j = \begin{cases} \alpha_f(a) + \alpha_f(a/p)t, & \text{if } p \mid a; \\ \alpha_f(a), & \text{if } p \nmid a; \end{cases}$$

holds for all $a \in \mathbb{N}_0$. This equality encodes a whole set of relations between eigenvalues and Fourier coefficients.

Andrianov proved a general decomposition theorem of type (0.2.1) in the context of Siegel modular forms [And77]. In this case the modular group $\mathrm{SL}_2(\mathbb{Z})$ is replaced by $\mathrm{Sp}_{2n}(\mathbb{Z})$ for some $n \in \mathbb{N}$, and one considers certain holomorphic functions on the Siegel upper half-space $\mathbb{H}_n$. The subgroup Γ_0 of upper triangular matrices is replaced by the "Siegel parabolic" in $\mathrm{Sp}_{2n}(\mathbb{Z})$, i. e. the subgroup of matrices whose lower left quadrant is zero.

It is then natural to ask if a decomposition of type (0.2.1) also holds for more general groups. Since this problem is of local nature, one may replace $\mathbb{Z}$ with the ring of integers $\mathcal{O}_F$ of a non-archimedean local field F. In this context, Gritsenko proved a decomposition theorem for $\mathrm{GL}_n(F)$ (all parabolics) [Gri88, Gri92] and for the classical groups $\mathrm{Sp}_{2n}(F)$, $\mathrm{SU}_n(F)$, and $\mathrm{SO}_n(F)$ (for the parabolics fixing a line in the standard representation) [Gri90].

The main result in [Gri92] found an application in the theory of p-adic L-functions, where it was recently used by Januszewski [Jan14] in order to define a projection map to obtain simultaneous eigenfunctions for certain Hecke operators. It is therefore reasonable to hope that a decomposition theorem for more general reductive groups will have applications in the theory of p-adic L-functions or p-adic modular forms.

In the second part of this thesis we generalize the theory developed by Andrianov in [And77] to a connected reductive F-group $\mathbf{G}$. Due to the generality we are forced to make some adjustments, so that the objects that occur in our context for, say, Sp_{2n} are not exactly the same as in [And77]. For example, there is no equivalent for "integral Hecke rings" (in the sense of [And77, p. 347]), which are defined for matrix groups only. We will give an overview of the theory and point out the differences along the way.

We fix a minimal parabolic subgroup $\mathbf{B}$ with Levi subgroup $\mathbf{Z}$ and unipotent radical $\mathbf{U}$. Let K be a special maximal parahoric subgroup of G (in good position relative to B). We write $K_X := K \cap X$ for any subset $X \subseteq G$. Let R be a commutative unital ring in which the characteristic p of the residue field of F is invertible. Fix a maximal standard parabolic $\mathbf{P}$ with standard Levi subgroup $\mathbf{M}$.

We work with an unnormalized version of the Satake homomorphism

$$\mathcal{S}_G \colon H_R(K,G) \longrightarrow H_R(K_Z,Z),$$

which was initially defined by Herzig [Her11a, Thm. 1.2] and studied in a more general context by Henniart and Vignéras [HV15]. The homomorphism $\mathcal{S}_G$ differs slightly from the spherical map that Andrianov uses, but the differences are negligible. The Satake homomorphism is injective and one can explicitly describe its image: it is the subalgebra of invariants under a certain twisted action of the finite Weyl group on $H_R(K_Z,Z)$.

Moreover, there is a commutative diagram

$$\begin{tikzcd} H_R(K_P, P) \arrow[r, "\Theta^P_M"] & H_R(K_M, M) \arrow[d, "\mathcal{S}_M"] \\ H_R(K, G) \arrow[u, hook] \arrow[r, "\mathcal{S}_G"'] & H_R(K_Z, Z), \end{tikzcd} \tag{0.2.2}$$

where the left vertical arrow is a natural embedding, by which we view $H_R(K,G)$ as a subalgebra of $H_R(K_P, P)$. As before, Θ^P_M is the homomorphism that is induced by the canonical projection map $P \twoheadrightarrow M$. The algebra $H_R(K_P, P)$ is non-commutative, but besides $H_R(K,G)$ it contains another commutative subalgebra of interest to us. We need to fix a strictly positive element $a_P \in M$. Then we consider the centralizer C^+_P of $(K_P a_P)$ in $H_R(K_P, P)$, i. e. the element corresponding to the double coset $K_P a_P = K_P a_P K_P$. It is commutative and independent of a_P. The following property of C^+_P is crucial: given any $X \in H_R(K_P, P)$, we have $(K_P a_P)^n X \in C^+_P$ for $n \gg 0$. This allows us to do computations in C^+_P rather than in $H_R(K_P, P)$. One would like to recover the element X from its projection into C^+_P, but this is not always possible. Indeed, elements in $\operatorname{Ker}\Theta^P_M$, which is also the set of all elements which are annihilated by some power of $(K_P a_P)$, cannot be recovered. We consider the R-submodule

$$\mathcal{O}^+_P := C^+_P . H_R(G,K) \subseteq H_R(K_P, P)$$

and try to prove that elements in $\mathcal{O}^+_P$ *can* be recovered from their projection into C^+_P. Following Andrianov's method, we construct a certain polynomial

$$\chi_{a_P}(t) \in H_R(K,G)[t],$$

such that $(\mathcal{S}_M \circ \Theta^P_M)(K_P a_P)$ is a root of the polynomial $\chi^{\mathcal{S}_G}_{a_P}(t)$ in $H_R(K_Z, Z)[t]$, where, given any polynomial $f(t) \in H_R(K,G)[t]$, we denote by $f^{\mathcal{S}_G}(t)$ the polynomial in $H_R(K_Z, Z)[t]$ that is obtained by applying $\mathcal{S}_G$ to the coefficients of $f(t)$. In general $\chi_{a_P}(t)$ differs slightly from the polynomial considered by Andrianov. Since $H_R(K_P, P)$ is a non-commutative algebra, we need to be careful with evaluating polynomials, as different ways to do this yield different results. Once we clarify, what it means to be a "left root" of a polynomial, we can formulate the following conjecture:

Conjecture 0.9. *The element $(K_P a_P)$ is a left root of $\chi_{a_P}(t)$.*

Despite being a concrete condition, it is in general very difficult to verify this conjecture. If $G = \mathrm{Sp}_{2n}(F)$ and P is the "Siegel parabolic", or if $G = \mathrm{GL}_n(F)$, then verifying the statement in Conjecture 0.9 is relatively easy. But already for the other classical groups considered in [Gri90] one needs a fairly explicit description of the images in $H_R(K_P, P)$ of some elements of $H_R(K,G)$. To do so with general connected reductive groups seems to be a hopeless endeavor. We were, however, able to pinpoint a condition on the maximal parabolic P under which we could verify the statement in Conjecture 0.9. Before describing our strategy to prove it, let us first show how to proceed in order to obtain a decomposition theorem of type (0.2.1).

Assume for now that P is a parabolic subgroup for which $(K_P a_P)$ is a left root of $\chi_{a_P}(t)$. Although $(K_P a_P)$ is not invertible in $H_R(K_P, P)$, one can use $\chi_{a_P}(t)$ to define "negative powers" of $(K_P a_P)$. They are denoted by $(K_P a_P)^{-n}$ for $n \in \mathbb{N}$ and are by definition contained in $\mathcal{O}_P^+$. The quotation marks already suggest that this is an abuse of notation. These negative powers are used to recover elements of $\mathcal{O}_P^+$ from their projection into C_P^+ as the next lemma shows:

Lemma 0.10. *For every $X \in \mathcal{O}_P^+$ we have*

$$(K_P a_P)^n \cdot X \cdot (K_P a_P)^{-n} = X, \quad \textit{for } n \gg 0.$$

With this tool available, it is easy to prove that the statement in Conjecture 0.9 is equivalent to the more general assertion that $\mathcal{O}_P^+$ is a complement of $\operatorname{Ker} \Theta_M^P$ in $H_R(K_P, P)$. The latter assertion is the key result to prove the following decomposition theorem (cf. also (0.2.2)):

Theorem 0.11. *Let $d(t) \in H_R(K, G)[t]$ be a polynomial. Assume that $d^{S_G}(t)$ decomposes in $H_R(K_Z, Z)[t]$ as*

$$d^{S_G}(t) = \tilde{f}(t) \cdot \tilde{g}(t),$$

where $\tilde{f}(t)$ has coefficients in $(S_M \circ \Theta_M^P)(C_P^+)$ and satisfies $\tilde{f}(0) = 1$. Then there exist polynomials $f(t)$ and $g(t)$ in $H_R(K_P, P)[t]$ lifting, respectively, $\tilde{f}(t)$ and $\tilde{g}(t)$ and having the same respective degrees, such that

$$d(t) = f(t) \cdot g(t) \quad \textit{in } H_R(K_P, P)[t].$$

We now specify a condition on the maximal parabolic P under which we can prove the statement in Conjecture 0.9, the condition being the following: the angle between any two roots in the unipotent radical U_P of P is non-obtuse. We call maximal parabolics with this property *non-obtuse*. One would like to know how many non-obtuse parabolics there are. Are there any? It turns out that all the parabolics that were considered by Andrianov and Gritsenko are non-obtuse. As there is a one-to-one correspondence between the maximal (standard) parabolics and elements of the basis (determined by B) of the relative root system Φ of G, it is possible to classify non-obtuse parabolics in terms of nodes in the Dynkin diagram. Then all nodes in type A_n correspond to non-obtuse parabolics, as do the extremal nodes in types B_n, C_n, and D_n. Moreover, we identify two in type E_6 and one in type E_7. However, there are none in types E_8, F_4, and G_2.

Assume that P is non-obtuse. There is a homomorphism $\nu\colon Z \to V$ into the $\mathbb{R}$-vector space V generated by the coroots of G (relative to a fixed maximal F-split torus of G inside Z). Denote by Z^- the preimage of the positive Weyl chamber under ν. We then reduce the statement in Conjecture 0.9 to the following assertion:

Theorem 0.12. *Let $a \in Z$ be strictly positive. Let $u \in U_P$, $z \in Z^-$, and $z' \in Z$ such that $uz' \in KzK$ and such that $\nu(a^{-1}) - \nu(z)$ is the sum of positive coroots. Then $az' \in M$ is positive and $aua^{-1} \in K_P$.*

This theorem might be interesting in its own right, for it contributes to the following problem: given an Iwasawa double coset $Uz'K$ and a Cartan double coset KzK, it is of general interest to study the intersection $Uz'K \cap KzK$. If $z \in Z^-$, it is well-known that this intersection is non-empty if and only if $\nu(z) - \nu(z')$ is a sum of positive coroots. However, very little is known about the $u \in U$ with $uz' \in KzK$. For p-adic Chevalley groups one step in this direction was made in [Dąb94] with K replaced by the standard Iwahori subgroup. In his article Dąbrowski obtains a description of the intersection between a version of Iwasawa double cosets and Iwahori double cosets in terms of certain "good subexpressions". By adapting the methods in the article of Lansky [Lan01] one might find a description of $Uz'K \cap KzK$ analogous to the one in [Dąb94]. However, since the results in [Dąb94] did not suffice to prove the statement in Conjecture 0.9, we did not pursue this line of research any further.

But what is the merit of Theorem 0.12 regarding the above problem? First, the requirement $u \in U_P$ is not a serious restriction when Φ is irreducible: assume $uz' \in KzK$ with $u \in U$. We may write $u = u_{U_P} u_M$ with $u_{U_P} \in U_P$ and $u_M \in U \cap M$. We find $k_1, k_2 \in K_M$ such that $k_1 u_M z' k_2 = z'' \in Z$. Consequently, we have $(k_1 u_{U_P} k_1^{-1}) \cdot z'' \in KzK$, and Theorem 0.12 provides information about $k_1 u_{U_P} k_1^{-1} \in U_P$, which should be very similar to u_{U_P}. Moreover, we have $u_M z' \in K_M z'' K_M$, and since M does contain at least one non-obtuse parabolic, we inductively obtain information about u_M from the theorem.

Let us now describe the quality of information that Theorem 0.12 provides if $u \in U_P$. Just like the local field F, the root groups U_α, for $\alpha \in \Phi$, are endowed with a certain valuation $\varphi_{0,\alpha}$. In fact, $\varphi_0 = (\varphi_{0,\alpha})_{\alpha \in \Phi}$ is the special point in the chosen apartment $\mathcal{A}$ that is used to define K. If we write

$$u = u_{\alpha_1} \cdots u_{\alpha_r} \tag{0.2.3}$$

with $u_{\alpha_i} \in U_{\alpha_i} \subseteq U_P$, then the theorem shows that the valuation of u_{α_i} is bounded below by $\langle \alpha_i, \nu(a) \rangle$.

Unfortunately, the proof of Theorem 0.12 is highly technical and involves a case-by-case analysis. The general idea, however, is not too complicated. Let $\mathcal{B}$ be the adjoint building of G. Then G acts by isometries on $\mathcal{B}$, revealing an abundance of information about G. We are mainly interested in the action of the root groups on $\mathcal{B}$. The root group element $u_\alpha \in U_\alpha \setminus \{1\}$ acts by fixing a half-apartment of $\mathcal{A}$ whose boundary coincides with the hyperplane $H := \left\{ x \in \mathcal{A} \,\middle|\, \langle \alpha, x - \varphi_0 \rangle + \varphi_{0,\alpha}(u_\alpha) = 0 \right\}$. In particular, we can read off the valuation of u_α from its action on $\mathcal{B}$. What is more, there exist unique elements $u'_{-\alpha}, u''_{-\alpha} \in U_{-\alpha}$, fixing the complementary half-apartment, such that $n_\alpha := u'_{-\alpha} u_\alpha u''_{-\alpha}$ acts on $\mathcal{A}$ as the reflection in H. This phenomenon is visualized in Figure 1.

We now describe the geometric intuition behind our approach to estimate the valuations of the u_{α_i} in (0.2.3). Let λ be the image of z and let μ be the image of z' in $\mathcal{A}$. There exists $k \in K$ such that $k.\lambda = u.\mu$. It is known that μ is contained inside the ball B_λ around φ_0 with radius $\|\lambda\|$ in $\mathcal{B}$, see Figure 2.

Assume that α_r is extremal in the cone generated by the roots in U_P and that the valuation of u_{α_r} is sufficiently negative. Using n_{α_r} as above, we may reflect μ along the

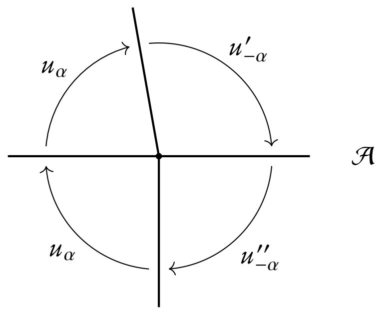

Figure 1: The action of $u'_{-\alpha} u_\alpha u''_{-\alpha}$ on $\mathcal{B}$.

Explanation: in this picture, the horizontal line denotes the apartment $\mathcal{A}$, the hyperplane H is represented by the marked point; u_α fixes the right half-apartment, and $u'_{-\alpha}$ and $u''_{-\alpha}$ fix the left half-apartment.

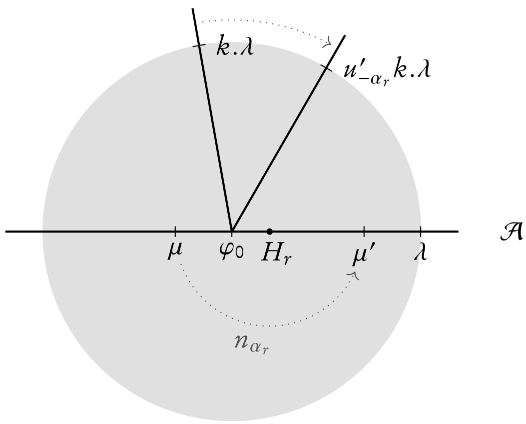

Figure 2: Measuring the valuation of u_{α_r}.

hyperplane H_r to obtain a new point $\mu' = n_{\alpha_r}.\mu$ in $\mathcal{A}$. We observe that μ' again lies inside B_λ. This forces the hyperplane H_r to meet B_λ, from which we obtain an effective estimate on the valuation of u_{α_r}. This step is the heart of an algorithm, which terminates when one of the reflected points μ' ends up on the boundary of B_λ.

One big problem, however, is that after the first step we potentially lose any information about the u_{α_i} for $i < r$. We therefore try to apply the first step of this algorithm to different decompositions of u. The reason why this eventually succeeds is exactly the non-obtuseness of P. This condition ensures that "most" roots are extremal in the cone generated by the roots in U_P. The remaining roots can be handled by a careful analysis.

1. Review of Bruhat-Tits theory

We start by reviewing those parts from Bruhat-Tits theory, that will be necessary in the following. The main references are the original papers of Bruhat and Tits [BT72, BT84]. For a large part we will follow [Vig16]. Apart from Proposition 1.9 nothing in this section is original work.

Let F be a local field with normalized valuation $\omega\colon F \to \mathbb{Z} \cup \{\infty\}$. We denote by $\mathcal{O}_F$, $\mathfrak{m}_F$, κ_F the valuation ring, its maximal ideal and the residue field, respectively. We fix a uniformizer π_F of F, i. e. an element $\pi_F \in \mathcal{O}_F$ with $\mathfrak{m}_F = \pi_F.\mathcal{O}_F$. The residue field κ_F is a finite field of characteristic $p > 0$, and we denote its cardinality by q.

In this thesis we denote algebraic groups by a boldface letter and their group of F-points by the corresponding lightface letter: given an algebraic group $\mathbf{H}$ over F, we denote by $H := \mathbf{H}(F)$ the group of F-rational points. The group H is always viewed as a topological group with respect to the topology induced from the π_F-adic one on F. We denote by $X^*(H)$, resp. $X_*(H)$, the abelian group of F*-characters*, resp. the set of F*-cocharacters* of $\mathbf{H}$, i. e. the F-morphisms of algebraic groups $\mathbf{H} \to \mathbb{G}_{\mathrm{m}}$, resp. $\mathbb{G}_{\mathrm{m}} \to \mathbf{H}$. If $\mathbf{H}$ is abelian, then $X_*(H)$ is also an abelian group. We denote by $\mathbf{H}^\circ$ the *connected component* and by $\mathscr{D}(\mathbf{H})$ the *derived subgroup* of $\mathbf{H}$.

1.1. The root system of a reductive group

Let $\mathbf{G}$ be a connected reductive group[3] over F. Let $\mathbf{T}$ be a maximal F-split torus of $\mathbf{G}$ with normalizer $\mathbf{N}_{\mathbf{G}}(\mathbf{T})$ and centralizer $\mathbf{Z}_{\mathbf{G}}(\mathbf{T})$. The group $W_0 := W(\mathbf{G}, \mathbf{T}) := N/Z$, where $N := \mathbf{N}_{\mathbf{G}}(\mathbf{T})(F)$ and $Z := \mathbf{Z}_{\mathbf{G}}(\mathbf{T})(F)$, is finite and called the *(finite) Weyl group* of $\mathbf{G}$. The adjoint action of T on the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ is diagonalizable, hence we obtain a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = \alpha_F(t) \cdot x \text{ for all } t \in T\}. \tag{1.1.1}$$

The set $\Phi := \Phi(\mathbf{G}, \mathbf{T}) := \{\alpha \in X^*(T) \mid \mathfrak{g}_\alpha \neq \{0\}\}$ is called the *(relative) root system* associated with $\mathbf{G}$ and $\mathbf{T}$. The elements of Φ are called the F*-roots* or *roots relative to* F of $\mathbf{G}$ (with respect to $\mathbf{T}$). Let N act on $X^*(T)$ via

$$(n.\alpha)_R(t) := \alpha_R(n^{-1}tn), \quad \text{for } n \in N,\ \alpha \in X^*(T),\ t \in \mathbf{T}(R), \text{ and } F\text{-algebras } R.$$

The adjoint action of N on $\mathfrak{g}$ permutes the eigenspaces via $n.\mathfrak{g}_\alpha = \mathfrak{g}_{n.\alpha}$ for $n \in N$, $\alpha \in X^*(T)$. As Z acts trivially on $X^*(T)$, we thus obtain an action of $W_0 = N/Z$ on the relative root system Φ. Similarly, the conjugation action of N on $\mathbf{T}$ induces an action of W_0 on $X_*(T)$.

Following [Bor91, §21], the subgroup $\mathbf{T}' := \big(\mathbf{T} \cap \mathscr{D}(\mathbf{G})\big)^\circ$ is a maximal F-split torus of $\mathscr{D}(\mathbf{G})$. If $\mathbf{T}_{\mathbf{C}}$ denotes the maximal F-split torus in the connected center $\mathbf{C}$ of $\mathbf{G}$, then we

[3] Here and in the following we always assume that $\mathbf{G}$ is *isotropic*, i. e. there exists a non-central F-split torus in $\mathbf{G}$ of positive dimension.

have $\mathbf{T} = \mathbf{T}' \cdot \mathbf{T}_\mathbf{C}$ and $\mathbf{T}' \cap \mathbf{T}_\mathbf{C}$ is finite. As $\mathbf{C}$ lies in the kernel of $\mathrm{Ad}\colon \mathbf{G} \to \mathrm{GL}_\mathfrak{g}$, the F-roots are trivial on $\mathbf{T}_\mathbf{C}$, hence the restriction map $X^*(T) \to X^*(T')$ embeds $\Phi(\mathbf{G},\mathbf{T})$ into $X^*(T')$. In fact, we thus obtain isomorphisms $\Phi(\mathbf{G},\mathbf{T}) \cong \Phi(\mathscr{D}(\mathbf{G}),\mathbf{T}')$ and $W(\mathbf{G},\mathbf{T}) \cong W(\mathscr{D}(\mathbf{G}),\mathbf{T}')$. Note that $X_*(\mathbf{T}_\mathbf{C}(F)) = X_*(C)$ and $X_*(T) = X_*(T') \times X_*(C)$. We fix a W_0-invariant scalar product $(\,\cdot\,,\cdot\,)$ on the finite-dimensional $\mathbb{R}$-vector space

$$V := X_*(T') \otimes_\mathbb{Z} \mathbb{R} = \big(X_*(T)/X_*(C)\big) \otimes_\mathbb{Z} \mathbb{R} \tag{1.1.2}$$

(this is possible, since W_0 is finite). The canonical pairing $\langle\,\cdot\,,\cdot\,\rangle\colon X^*(T') \times X_*(T') \to \mathbb{Z}$ given by $\langle \chi, \lambda\rangle = \chi \circ \lambda$ under the identification $\mathrm{End}(\mathbb{G}_\mathrm{m}) \cong \mathbb{Z}$, is perfect, i. e. it induces isomorphisms $X^*(T') \cong \mathrm{Hom}_\mathbb{Z}(X_*(T'),\mathbb{Z})$ and $X_*(T') \cong \mathrm{Hom}_\mathbb{Z}(X^*(T'),\mathbb{Z})$. Consequently, we may identify $X^*(T') \otimes_\mathbb{Z} \mathbb{R}$ with the $\mathbb{R}$-linear dual $V^* = \mathrm{Hom}_\mathbb{R}(V,\mathbb{R})$ of V. Via the isomorphism $V \to V^*$, $v \mapsto (v,\cdot\,)$, we endow V^* with a W_0-invariant scalar product.

Theorem 1.1. *The set $\Phi = \Phi(\mathbf{G},\mathbf{T})$ is a root system in $(V^*, (\,\cdot\,,\cdot\,))$ with Weyl group $W_0 = N/Z$, i. e.*[4] *it satisfies the following conditions:*

(R_I) Φ is finite, spans V^* and does not contain 0*.*

(R_{II}) For each $\alpha \in \Phi$ there exists a (unique) element $\alpha^\vee \in V^{**} = V$ with $\langle \alpha, \alpha^\vee\rangle = 2$ and such that the reflection

$$s_{\alpha,\alpha^\vee}\colon V^* \longrightarrow V^*, \quad x \longmapsto x - \langle x, \alpha^\vee\rangle \cdot \alpha \tag{1.1.3}$$

leaves Φ invariant. We have $\alpha^\vee = \frac{2(\alpha,\cdot)}{(\alpha,\alpha)}$ and $\alpha^\vee \in X_*(T')$.

(R_{III}) For each $\alpha \in \Phi$ we have $\langle \Phi, \alpha^\vee\rangle \subseteq \mathbb{Z}$.

(R_{IV}) The reflections $s_{\alpha,\alpha^\vee}$, for $\alpha \in \Phi$, generate the group W_0.

The set $\Phi^\vee := \Phi^\vee(\mathbf{G},\mathbf{T}) := \{\alpha^\vee \in X_*(T') \,|\, \alpha \in \Phi(\mathbf{G},\mathbf{T})\}$ is a root system with Weyl group W_0.

Proof. See [Bor91, Theorems 21.2 and 21.6]. The last assertion is immediate. □

Remark 1.2. (a) A root system Ψ is called *reduced* if $\alpha \in \Psi$ implies $2\alpha \notin \Psi$. Given a root system Ψ, the set

$$\Psi_{\mathrm{red}} := \{\alpha \in \Psi \,|\, \tfrac{\alpha}{2} \notin \Psi\} \tag{1.1.4}$$

is a reduced root system with the same Weyl group as Ψ.

If $\mathbf{G}$ is F-split, the associated root system $\Phi(\mathbf{G},\mathbf{T})$ is reduced. However, this need not be the case if $\mathbf{G}$ is not F-split.

(b) Given $\mathbb{R}$-vector spaces E_i and root systems $\Psi_i \subseteq E_i$, $i = 1,\dots,n$, the set $\Psi := \bigcup_{i=1}^n \Psi_i$ is a root system in $E := \bigoplus_{i=1}^n E_i$, called the *direct sum* of the root systems $\Psi_1,\dots,\Psi_n$. A root system is called *irreducible* if it is non-empty and not a direct sum of two non-empty root systems. Each root system can be uniquely decomposed into irreducible root systems [Bou81, Ch. VI, §1.2, Prop. 6].

[4] The notions *root system* and *Weyl group* are to be understood as in [Bou81, Ch. VI, §1.1] which a posteriori justifies the nomenclature above.

We may thus decompose $V = V_1 \oplus \cdots \oplus V_n$ such that, after identifying $V^* = V_1^* \oplus \cdots \oplus V_n^*$, $\Phi_i := \Phi(\mathbf{G}, \mathbf{T}) \cap V_i^*$ is an irreducible root system in V_i^*, $i = 1, \ldots, n$, and $\Phi(\mathbf{G}, \mathbf{T}) = \bigcup_{i=1}^n \Phi_i$. Then also $\Phi_i^\vee := \Phi^\vee(\mathbf{G}, \mathbf{T}) \cap V_i$ is irreducible with $\Phi^\vee(\mathbf{G}, \mathbf{T}) = \bigcup_{i=1}^n \Phi_i^\vee$.

Note that for $\alpha \in \Phi_i$ the reflection $s_{\alpha,\alpha^\vee}$ acts trivially on V_j and V_j^* for $j \neq i$. This gives a decomposition $W_0 = W_{0,1} \times \cdots \times W_{0,n}$, where $W_{0,i}$ is the Weyl group of Φ_i, $i = 1, \ldots, n$.

(c) Given a root system Ψ inside an $\mathbb{R}$-vector space E, there exists a subset $\Delta \subseteq \Psi$ satisfying the following conditions:

1. Δ is a basis of E;
2. given $\alpha \in \Psi$ and writing $\alpha = \sum_{\beta \in \Delta} \lambda_\beta \beta$, we have $\lambda_\beta \in \mathbb{Z}_{\geq 0}$ for *all* $\beta \in \Delta$ or $\lambda_\beta \in \mathbb{Z}_{\leq 0}$ for *all* $\beta \in \Delta$.

The subset Δ is called a *basis* of Ψ. The Weyl group acts simply transitively on the set of bases of Ψ [Bou81, Ch. VI, §1.5, Thm. 2 and §1.6, Thm. 3]. Thus, writing Ψ^+ for the set of $\alpha = \sum_{\beta \in \Delta} \lambda_\beta \beta \in \Psi$ with all $\lambda_\beta \in \mathbb{Z}_{\geq 0}$, and $\Psi^- := -\Psi^+$, we have $\Psi = \Psi^+ \sqcup \Psi^-$. The elements in Δ are called *simple roots* and those in Ψ^+ (resp. Ψ^-) are called *positive roots* (resp. *negative roots*).

There exists a linear form $\lambda \in E^*$ such that $\Psi^+ = \{\alpha \in \Psi \mid \lambda(\alpha) > 0\}$. Conversely, given any linear form $\lambda \in E^*$ with $0 \notin \lambda(\Psi)$ there exists a unique basis $\Delta \subseteq \Psi$ such that $\{\alpha \in \Psi \mid \lambda(\alpha) > 0\}$ is the associated set of positive roots.

1.2. The root group datum of a group

Let V be an $\mathbb{R}$-vector space and $\Phi \subseteq V^*$ be a root system. We fix a basis $\Delta \subseteq \Phi$ and hence a choice of positive roots Φ^+ inside Φ.

Definition 1.3. [BT72, (6.1.1)] Let G be a group. A datum $(Z, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ is called a *root group datum of type* Φ if it satisfies the following conditions:

(DR$_1$) Z and $U_\alpha \neq \{1\}$ are subgroups of G, for all $\alpha \in \Phi$.

(DR$_2$) For each $\alpha, \beta \in \Phi$ the commutator subgroup $[U_\alpha, U_\beta]$ is contained in the group generated by the $U_{r\alpha + s\beta}$ for $r, s \in \mathbb{N}$ with $r\alpha + s\beta \in \Phi$.

(DR$_3$) If $\alpha, 2\alpha \in \Phi$ then we have $U_{2\alpha} \subsetneq U_\alpha$.

(DR$_4$) For each $\alpha \in \Phi$ the set $M_\alpha \subseteq G$ is a right coset under Z and we have $U_{-\alpha}^* \subseteq U_\alpha M_\alpha U_\alpha$, where we put $U_\alpha^* := U_\alpha \setminus \{1\}$ for $\alpha \in \Phi$.

(DR$_5$) For each $\alpha, \beta \in \Phi$ and $n \in M_\alpha$ we have $n U_\beta n^{-1} = U_{s_{\alpha,\alpha^\vee}(\beta)}$.

(DR$_6$) If U^+ (resp. U^-) denotes the group generated by all U_α with $\alpha \in \Phi^+$ (resp. $\alpha \in -\Phi^+$), then we have $ZU^+ \cap U^- = \{1\}$.

The root group datum $\big(Z, (U_\alpha, M_\alpha)_{\alpha \in \Phi}\big)$ is said to be *generating* if G is generated by Z and U_α, for $\alpha \in \Phi$.

Remark 1.4. The following consequences of the above axioms prove to be very useful.

(a) If $2\alpha \notin \Phi$ we put $U_{2\alpha} = \{1\}$. In general, it follows from axiom (DR$_2$) that the commutator subgroup $[U_\alpha, U_\alpha]$ is contained in $U_{2\alpha}$, and that $U_{2\alpha}$ is central in U_α.

(b) [BT72, (6.1.2) (2)] Let $\alpha \in \Phi$ and $u \in U_{-\alpha}^*$. There exists a unique element $m(u) \in M_\alpha$ such that $u \in U_\alpha m(u) U_\alpha$. More precisely, there exists a unique triple $(u', m, u'') \in U_\alpha \times G \times U_\alpha$ such that $u = u'mu''$, $mU_\alpha m^{-1} = U_{-\alpha}$ and $mU_{-\alpha}m^{-1} = U_\alpha$; we then have $m \in M_\alpha$ and $u' \neq 1$.

(c) [BT72, (6.1.2) (9)] Let $\alpha \in \Phi$ and L_α be the subgroup of G generated by Z, U_α and $U_{-\alpha}$. Then we have

$$M_\alpha = \left\{ x \in L_\alpha \,\middle|\, xU_\alpha x^{-1} = U_{-\alpha} \text{ and } xU_{-\alpha}x^{-1} = U_\alpha \right\}. \tag{1.2.1}$$

In particular, M_α is completely determined by Z, U_α and $U_{-\alpha}$. It thus makes sense to speak of the root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$.

(d) Let N be the subgroup of G generated by Z and the $m(U_{-\alpha}^*) \subseteq M_\alpha$ for $\alpha \in \Phi$. From (DR5) it follows that there exists a unique epimorphism

$$^{\mathrm{v}}\nu \colon N \longrightarrow W_0, \tag{1.2.2}$$

where W_0 is the Weyl group of Φ, such that for all $\alpha \in \Phi$ and all $n \in N$ we have $nU_\alpha n^{-1} = U_{^{\mathrm{v}}\nu(n)(\alpha)}$. Moreover, we have $^{\mathrm{v}}\nu(M_\alpha) = \{s_{\alpha,\alpha^\vee}\}$ for all $\alpha \in \Phi$. The kernel of $^{\mathrm{v}}\nu$ coincides with $Z = N \cap ZU^+$ [BT72, Cor. (6.1.11) (ii)].

(e) [BT72, (6.1.2) (11)] The group N normalizes Z. Since W_0 acts transitively on the set of bases of Φ, we see that (DR6) remains valid if we replace Φ^+ by a different choice of positive roots.

Example 1.5. (a) Consider the general linear group $\mathrm{GL}_n(F)$. We obtain a root group datum as follows: Denote by E_{ij} the elementary matrix, whose only non-zero entry is a 1 in the (i,j)-th spot. Let E_n be the $n \times n$-identity matrix. For Z we take the subgroup of diagonal matrices $\left\{\sum_{i=1}^n a_i E_{ii} \,\middle|\, a_i \in F^\times\right\}$ of $\mathrm{GL}_n(F)$. We consider the root system $\Phi \subseteq \mathbb{R}^n$ of type A_{n-1} given by

$$\Phi := \left\{ \alpha_{ij} := e_i - e_j \,\middle|\, 1 \leq i \neq j \leq n \right\},$$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$. If we denote by $\chi_1, \ldots, \chi_n \in (\mathbb{R}^n)^*$ its dual basis, we find that the coroot system is given by

$$\Phi^\vee := \left\{ \alpha_{ij}^\vee := \chi_i - \chi_j \,\middle|\, 1 \leq i \neq j \leq n \right\}.$$

Given $\alpha_{ij}, \alpha_{kl} \in \Phi$, we then compute

$$s_{\alpha_{ij}}(\alpha_{kl}) = \alpha_{kl} - \langle \alpha_{kl}, \alpha_{ij}^\vee \rangle \cdot \alpha_{ij} = \alpha_{\tau_{ij}(k), \tau_{ij}(l)},$$

where τ_{ij} is the transposition on $\{1, 2, \ldots, n\}$ interchanging i and j. We fix the basis $\Delta := \left\{\alpha_{i,i+1} \,\middle|\, 1 \leq i \leq n-1\right\}$ of Φ. Then $\Phi^+ = \left\{\alpha_{ij} \in \Phi \,\middle|\, 1 \leq i < j \leq n\right\}$ is the set of positive roots. Set $U_{\alpha_{ij}} := \left\{E_n + \lambda E_{ij} \,\middle|\, \lambda \in F\right\}$ for $\alpha_{ij} \in \Phi$. Finally, put

$$M_{\alpha_{ij}} := \left\{ \sum_{r \neq i,j} a_r E_{rr} + a_i E_{ij} + a_j E_{ji} \,\middle|\, a_r \in F^\times \text{ for } 1 \leq r \leq n \right\} \quad \text{for } i \neq j.$$

We claim that $\big(Z,(U_\alpha,M_\alpha)_{\alpha\in\Phi}\big)$ is a generating root group datum of $\mathrm{GL}_n(F)$. There is nothing to say about (DR$_1$) and (DR$_3$). In order to verify (DR$_2$) it suffices to consider $\alpha,\beta\in\Phi$ with $\beta\neq\pm\alpha$. Given $\alpha_{ij},\alpha_{kl}\in\Phi$, we thus may assume $(i,j)\neq(l,k)$. A direct computation then shows

$$\begin{aligned}(E_n+\lambda E_{ij})(E_n+\lambda' E_{kl})(E_n-\lambda E_{ij})(E_n-\lambda' E_{kl})&\\ &= E_n+\lambda\lambda'\delta_{jk}E_{il}-\lambda\lambda'\delta_{li}E_{kj}\\ &=(E_n+\lambda\lambda'\delta_{jk}E_{il})\cdot(E_n-\lambda\lambda'\delta_{li}E_{kj}) \qquad (1.2.3)\end{aligned}$$

for $\lambda,\lambda'\in F$ (where δ_{ij} is the Kronecker symbol). Notice that $j=k$ implies $\alpha_{ij}+\alpha_{kl}=\alpha_{il}\in\Phi$ and $l=i$ implies $\alpha_{ij}+\alpha_{kl}=\alpha_{kj}\in\Phi$. In either case the above computation shows that the commutator subgroup $[U_{\alpha_{ij}},U_{\alpha_{kl}}]$ is contained in the subgroup of $\mathrm{GL}_n(F)$ generated by $U_{r\alpha_{ij}+s\alpha_{kl}}$ for $r,s\in\mathbb{N}$ with $r\alpha_{ij}+s\alpha_{kl}\in\Phi$, whence (DR$_2$). As U^+ (resp. U^-) is the group of upper (resp. lower) triangular matrices with all diagonal entries equal to 1, axiom (DR$_6$) is visibly true. Let $\alpha_{ij}\in\Phi$. It is clear that $M_{\alpha_{ij}}$ is a right coset under Z. Moreover, for $x\in F^\times$, the computation

$$\begin{pmatrix}1&0\\x&1\end{pmatrix}=\begin{pmatrix}1&x^{-1}\\0&1\end{pmatrix}\begin{pmatrix}0&-x^{-1}\\x&0\end{pmatrix}\begin{pmatrix}1&x^{-1}\\0&1\end{pmatrix}$$

generalizes and shows $U^*_{-\alpha_{ij}}\subseteq U_{\alpha_{ij}}M_{\alpha_{ij}}U_{\alpha_{ij}}$, i. e. (DR$_4$). It remains to show (DR$_5$). Let $\alpha_{ij},\alpha_{kl}\in\Phi$ and consider the element $P_{ij}:=\sum_{r\neq i,j}E_{rr}+E_{ij}+E_{ji}\in M_{\alpha_{ij}}$. As left (resp. right) multiplication with $P_{ij}=P_{ij}^{-1}$ interchanges the i-th and j-th row (resp. column), it follows that $P_{ij}U_{\alpha_{kl}}P_{ij}^{-1}=U_{\alpha_{\tau_{ij}(k),\tau_{ij}(l)}}=U_{s_{\alpha_{ij}}(\alpha_{kl})}$. Since each U_α, for $\alpha\in\Phi$, is normalized by Z, this shows that (DR$_5$) holds. Thus, $\big(Z,(U_\alpha,M_\alpha)_{\alpha\in\Phi}\big)$ is a root group datum of type A_{n-1}. It is generating by Gaussian elimination. We note that the group N in Remark 1.4, (d) is the group of monomial matrices in $\mathrm{GL}_n(F)$, which is also the normalizer of Z. The Weyl group $W_0=N/Z$ is the symmetric group $\mathfrak{S}_n$.

(b) Given a connected reductive group $\mathbf{G}$ over F, we fix a maximal F-split torus $\mathbf{T}$. Let $\Phi=\Phi(\mathbf{G},\mathbf{T})$ be the root system attached to $\mathbf{G}$ (see 1.1). Let Z be group of F-points of the centralizer $\mathbf{Z}_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. Given $\alpha\in\Phi$, there exists a unique closed connected unipotent F-subgroup $\mathbf{U}_\alpha$, which is normalized by $\mathbf{Z}_{\mathbf{G}}(\mathbf{T})$ and has Lie algebra $\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ (in the notation of section 1.1) [Bor91, 21.9 Prop.]. Then $\big(Z,(U_\alpha)_{\alpha\in\Phi}\big)$ is a generating root group datum of type Φ [BT84, 1.1.13].

We recall the following lemma [BT72, Lem. (6.1.7)].

Lemma 1.6. *Let X be a group. Let $\Psi\subseteq\Phi^+$ be a subset and let $\Psi_{\mathrm{red}}=\left\{\alpha\in\Psi\,\middle|\,\frac{\alpha}{2}\notin\Psi\right\}$. For each $\alpha\in\Psi$ let Y_α be a subgroup of X; we put $X_\alpha:=Y_\alpha$ and $X_{2\alpha}=\{1\}$ if $2\alpha\notin\Psi$, and $X_\alpha:=Y_\alpha Y_{2\alpha}$ if $2\alpha\in\Psi$. Suppose further that the following conditions are satisfied:*

(i) X is generated by $\bigcup_{\alpha\in\Psi}Y_\alpha$;

(ii) for all $\alpha,\beta\in\Psi$ the commutator subgroup $[Y_\alpha,Y_\beta]$ is contained in the subgroup of X generated by the $Y_{r\alpha+s\beta}$, for $r,s\in\mathbb{N}$ with $r\alpha+s\beta\in\Psi$;

(iii) *the intersection of the groups generated by* $\bigcup_{\substack{\alpha\in\Psi \\ \langle\alpha,v\rangle\le 0}} Y_\alpha$ *and* $\bigcup_{\substack{\alpha\in\Psi \\ \langle\alpha,v\rangle> 0}} Y_\alpha$*, respectively, is trivial for each* $v \in V$.

Then X *is a nilpotent group and the map* $\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \to X$*, given by multiplication, is bijective with respect to any choice of ordering of the factors.*

Definition 1.7. Let $\Psi \subseteq \Phi^+$ and $X = \langle X_\alpha \mid \alpha \in \Psi\rangle$ be as in Lemma 1.6.

(a) We define a *partial order* on Ψ as follows: given $\alpha, \beta \in \Psi$, we put $\alpha \le \beta$ if there exist $\gamma_1, \dots, \gamma_n \in \Psi$ and $r \in \mathbb{N}$, $s_1, \dots, s_n \in \mathbb{Z}_{\ge 0}$ with $\beta = r\alpha + \sum_{i=1}^n s_i\gamma_i$. It is clear that this relation is reflexive and transitive. Given $v \in V$ with $\langle\alpha, v\rangle > 0$ for all $\alpha \in \Phi^+$, we see that $\alpha \le \beta$ implies $\langle\alpha, v\rangle \le \langle\beta, v\rangle$ with equality if and only if $\alpha = \beta$. Hence the relation is also antisymmetric.

(b) By an *ordering of the factors* of $\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha$ we mean a bijection $o\colon \Psi_{\mathrm{red}} \to \{1, 2, \dots, |\Psi_{\mathrm{red}}|\}$ such that the map $\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \to X$ at the end of Lemma 1.6 is given by

$$(x_\alpha)_{\alpha\in\Psi_{\mathrm{red}}} \longmapsto \prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha \coloneqq x_{o^{-1}(1)} \cdot x_{o^{-1}(2)} \cdots x_{o^{-1}(|\Psi_{\mathrm{red}}|)}.$$

Lemma 1.8. *Let* $\Psi \subseteq \Phi^+$, Ψ_{red}, X *and* $(X_\alpha)_{\alpha\in\Psi_{\mathrm{red}}}$ *be as in Lemma 1.6, satisfying (i), (ii), and (iii) above. Let* $f : X \to X$ *be a group homomorphism such that*

$$f(x_\alpha)x_\alpha^{-1} \in \langle X_\beta \mid \beta > \alpha\rangle \quad \textit{for all } x_\alpha \in X_\alpha, \textit{all } \alpha \in \Psi_{\mathrm{red}}.$$

Fix an ordering $o\colon \Psi_{\mathrm{red}} \to \{1, 2, \dots, |\Psi_{\mathrm{red}}|\}$ *of the factors. For all* $\alpha \in \Psi_{\mathrm{red}}$ *and all* $(x_\beta)_{\beta<\alpha} \in \prod_{\substack{\beta\in\Psi_{\mathrm{red}} \\ \beta<\alpha}} X_\beta$ *there exists a unique element* $z_\alpha \in X_\alpha$ *and a unique group homomorphism* $\tilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ *factoring through* $X_\alpha/X_{2\alpha}$ *such that*

$$f\Bigl(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\Bigr) = \prod_{\alpha\in\Psi_{\mathrm{red}}} \bigl(z_\alpha\tilde{z}_\alpha(x_\alpha)x_\alpha\bigr) \quad \textit{for all } x_\alpha \in X_\alpha, \textit{all } \alpha \in \Psi_{\mathrm{red}}. \tag{1.2.4}$$

Proof. We remark that (ii) in Lemma 1.6 implies that $X_{2\alpha}$ is central in X_α, and that the commutator subgroup $[X_\alpha, X_\alpha]$ is contained in $X_{2\alpha}$.

We induct on $|\Psi_{\mathrm{red}}|$. Suppose first that $\Psi_{\mathrm{red}} = \{\alpha\}$. The hypothesis on f implies $\tilde{z}_\alpha(x) \coloneqq f(x)x^{-1} \in X_{2\alpha}$ for $x \in X_\alpha$ and $\tilde{z}_\alpha(X_{2\alpha}) = \{1\}$. We show that $\tilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ is a group homomorphism. Given $x, y \in X_\alpha$, we compute

$$\begin{aligned}\tilde{z}_\alpha(xy) &= f(xy)\cdot(xy)^{-1} = f(x)f(y)y^{-1}x^{-1} \\ &= f(x)\tilde{z}_\alpha(y)x^{-1} = f(x)x^{-1}\cdot\tilde{z}_\alpha(y) = \tilde{z}_\alpha(x)\cdot\tilde{z}_\alpha(y).\end{aligned}$$

This proves the base case (with $z_\alpha \coloneqq 1$). Now suppose $|\Psi_{\mathrm{red}}| > 1$ and choose a maximal $\alpha_0 \in \Psi_{\mathrm{red}}$ (recall Definition 1.7 (a)). We start with the following claim:

Claim. *Suppose* $f(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha$ *for* $(x_\alpha)_\alpha, (y_\alpha)_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha$*. Then* y_{α_0} *depends only on the* x_β *with* $\beta \le \alpha_0$.

Proof of the claim. Consider $\Psi' := \{\beta \in \Psi_{\mathrm{red}} \mid \beta \not\leq \alpha_0\}$ and let Z_{α_0} be the subgroup of X generated by $\bigcup_{\beta\in\Psi'} X_\beta$. Notice that $\beta \in \Psi'$, $\gamma \in \Psi_{\mathrm{red}}$ and $\gamma \geq \beta$ imply $\gamma \in \Psi'$, i. e. Ψ' is an *upper subset* of Ψ_{red}. Therefore, the hypotheses of Lemma 1.6 remain satisfied for Z_{α_0}, Ψ' and $(X_\alpha)_{\alpha\in\Psi'}$ instead of X, Ψ and $(Y_\alpha)_{\alpha\in\Psi}$. Hence, the multiplication map $\prod_{\alpha\in\Psi'} X_\alpha \to Z_{\alpha_0}$ is bijective. Moreover, condition (ii) implies that Z_{α_0} is a normal subgroup of X, since Ψ' is an upper subset of Ψ_{red}. Consequently, the canonical projection map $\mathrm{pr}_{\leq\alpha_0}\colon \prod_{\beta\in\Psi_{\mathrm{red}}} X_\beta \to \prod_{\beta\leq\alpha_0} X_\beta$ is a group homomorphism. The hypothesis on f implies $f(Z_{\alpha_0}) \subseteq Z_{\alpha_0}$, since Ψ' is an upper subset of Ψ_{red}. We obtain an induced group homomorphism $\overline{f}\colon X/Z_{\alpha_0} \to X/Z_{\alpha_0}$ such that, after identifying $X \cong \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha$ and $X/Z_{\alpha_0} \cong \prod_{\alpha\leq\alpha_0} X_\alpha$, the following diagram commutes:

$$\begin{tikzcd}
\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{\leq\alpha_0}"'] & \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \arrow[d, "\mathrm{pr}_{\leq\alpha_0}"] \\
\prod_{\alpha\leq\alpha_0} X_\alpha \arrow[r, "\overline{f}"'] & \prod_{\alpha\leq\alpha_0} X_\alpha .
\end{tikzcd}$$

From this it is immediate that y_{α_0} only depends on the x_β with $\beta \leq \alpha_0$. The claim is proved. □

Let $(x_\alpha)_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}}$ and write $f(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha$ for certain $y_\alpha \in X_\alpha$, $\alpha \in \Psi_{\mathrm{red}}$. We prove the presentation (1.2.4) in two steps.

Step 1: We prove $y_\alpha = z_\alpha \tilde{z}_\alpha(x_\alpha)x_\alpha$ for $\alpha \neq \alpha_0$ with z_α and $\tilde{z}_\alpha$ as in the statement of the lemma. This follows from the induction hypothesis as follows: Recall that X_{α_0} is normal in X, hence the quotient $X' := X/X_{\alpha_0}$ is a group. Put $\Psi' := \Psi \smallsetminus \{\alpha_0, 2\alpha_0\}$ and $\Psi'_{\mathrm{red}} = \left\{\alpha \in \Psi' \,\middle|\, \frac{\alpha}{2} \notin \Psi'\right\}$. Under the projection map $X \twoheadrightarrow X'$ the subgroups Y_α, X_α of X embed into X' for $\alpha \in \Psi'$. By the hypothesis on f we obtain an induced homomorphism $f'\colon X' \to X'$. The hypotheses of the lemma remain satisfied if we replace X, Ψ, $(Y_\alpha)_{\alpha\in\Psi}$, f by X', Ψ', $(Y_\alpha)_{\alpha\in\Psi'_{\mathrm{red}}}$, f'. Notice that the diagram

$$\begin{tikzcd}
X = \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha \arrow[r, "f"] \arrow[d, "\mathrm{pr}"'] & \prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha = X \arrow[d, "\mathrm{pr}"] \\
X' = \prod_{\alpha\in\Psi'_{\mathrm{red}}} X_\alpha \arrow[r, "\overline{f}"'] & \prod_{\alpha\in\Psi'_{\mathrm{red}}} X_\alpha = X'
\end{tikzcd}$$

commutes. Therefore, we have $\overline{f}(\prod_{\alpha\in\Psi'_{\mathrm{red}}} x_\alpha) = \prod_{\alpha\in\Psi'_{\mathrm{red}}} y_\alpha$ and the induction hypothesis implies $y_\alpha = z_\alpha \tilde{z}_\alpha(x_\alpha)x_\alpha$ for certain elements $z_\alpha \in X_\alpha$ and group homomorphisms $\tilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ factoring through $X_\alpha/X_{2\alpha}$ and only depending on the x_β with $\beta < \alpha$, $\alpha \in \Psi'_{\mathrm{red}} = \Psi_{\mathrm{red}} \smallsetminus \{\alpha_0\}$. This establishes Step 1.

Step 2: We prove $y_{\alpha_0} = z_{\alpha_0}\tilde{z}_{\alpha_0}(x_{\alpha_0})x_{\alpha_0}$ with z_{α_0} and $\tilde{z}_{\alpha_0}$ as in the statement of the lemma. We introduce the following notation:

$$x^< := \prod_{\substack{\alpha\in\Psi_{\mathrm{red}} \\ o(\alpha)<o(\alpha_0)}} x_\alpha, \qquad x^> := \prod_{\substack{\alpha\in\Psi_{\mathrm{red}} \\ o(\alpha)>o(\alpha_0)}} x_\alpha, \qquad x := x^< x_{\alpha_0} x^>, \qquad x' := x^< x^>,$$

$$f(x^<) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha^<, \qquad f(x^>) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha^>, \qquad f(x) = \prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha, \qquad f(x') = \prod_{\alpha\in\Psi_{\mathrm{red}}} y'_\alpha .$$

From the claim we deduce that $y_{\alpha_0}^{<}$ (resp. $y_{\alpha_0}^{>}$, resp. y'_{α_0}) depends only on the x_β with $\beta < \alpha_0$ and $o(\beta) < o(\alpha_0)$ (resp. $o(\beta) > o(\alpha_0)$, resp. $o(\beta) \neq o(\alpha_0)$). Recall that $X_{2\alpha_0}$ is central in X and that X_{α_0} is centralized by all X_β with $\beta \neq \alpha_0$. Using this we compute

$$\begin{aligned}\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha = f(x) &= f(x^<)\cdot f(x_{\alpha_0})\cdot f(x^>) = \Bigg(\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha^<\Bigg)\cdot f(x_{\alpha_0})\cdot\Bigg(\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha^>\Bigg)\\ &= \Bigg(\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha^<\Bigg)\cdot\Bigg(\prod_{\alpha\in\Psi_{\mathrm{red}}} y_\alpha^>\Bigg)\cdot[f(x_{\alpha_0}),y_{\alpha_0}^>]\cdot f(x_{\alpha_0})\\ &= f(x^<)\cdot f(x^>)\cdot[f(x_{\alpha_0}),y_{\alpha_0}^>]\cdot f(x_{\alpha_0})\\ &= f(x')\cdot[f(x_{\alpha_0}),y_{\alpha_0}^>]\cdot f(x_{\alpha_0}) = \Bigg(\prod_{\alpha\in\Psi_{\mathrm{red}}} y'_\alpha\Bigg)\cdot[f(x_{\alpha_0}),y_{\alpha_0}^>]\cdot f(x_{\alpha_0}).\end{aligned}$$

We obtain $y_{\alpha_0} = y'_{\alpha_0}\cdot[f(x_{\alpha_0}),y_{\alpha_0}^>]\cdot f(x_{\alpha_0})$. As was mentioned above the element $z_{\alpha_0} := y'_{\alpha_0} \in X_\alpha$ only depends on the x_β with $\beta < \alpha_0$. We put

$$\tilde{z}_{\alpha_0}(x_{\alpha_0}) := [f(x_{\alpha_0}),y_{\alpha_0}^>]\cdot f(x_{\alpha_0})x_{\alpha_0}^{-1} \in X_{2\alpha_0}. \tag{1.2.5}$$

As $X_{2\alpha_0}$ is central in X_{α_0} and f is, by hypothesis, the identity on $X_{2\alpha_0}$, we have $\tilde{z}_{\alpha_0}(X_{2\alpha_0}) = \{1\}$. It remains to show that this defines a group homomorphism $\tilde{z}_{\alpha_0}\colon X_{\alpha_0} \to X_{2\alpha_0}$. The base case shows that $X_{\alpha_0} \to X_{2\alpha_0}$, $x \mapsto f(x)x^{-1}$ is a homomorphism. As $X_{2\alpha_0}$ is abelian it suffices to show that $X_{\alpha_0} \to X_{2\alpha_0}$, $x \mapsto [f(x),y_{\alpha_0}^>]$ is a homomorphism. This is immediate once we notice that $[uv,w] = u[v,w]u^{-1}\cdot[u,w]$ for all $u,v,w \in X_{\alpha_0}$. We conclude $y_{\alpha_0} = z_{\alpha_0}\tilde{z}_{\alpha_0}(x_{\alpha_0})x_{\alpha_0}$, with z_{α_0} and $\tilde{z}_{\alpha_0}$ only depending on the x_β with $\beta < \alpha_0$. This establishes Step 2.

Steps 1 and 2 together yield the presentation (1.2.4). This finishes the proof. □

For later use we rephrase Lemma 1.8 in the context of groups with a root group datum.

Proposition 1.9. *Let G be a group with root group datum $\big(Z,(U_\alpha)_{\alpha\in\Phi}\big)$ (cf. Remark 1.4 (c)). Let $\Psi \subseteq \Phi^+$ be a subset and let $\Psi_{\mathrm{red}} = \big\{\alpha \in \Psi \,\big|\, \frac{\alpha}{2} \notin \Psi\big\}$. For each $\alpha \in \Psi$ let Y_α be a subgroup of U_α; we put $X_\alpha := Y_\alpha$ and $X_{2\alpha} = \{1\}$ if $2\alpha \notin \Psi$, and $X_\alpha := Y_\alpha Y_{2\alpha}$ if $2\alpha \in \Psi$. Let X be the subgroup of U^+ generated by $\bigcup_{\alpha\in\Psi} Y_\alpha$. Suppose that condition (ii) of Lemma 1.6 is satisfied, i. e.*

(ii) for all $\alpha,\beta \in \Psi$ the commutator subgroup $[Y_\alpha, Y_\beta]$ is contained in the subgroup of X generated by the $Y_{r\alpha+s\beta}$, for $r,s \in \mathbb{N}$ with $r\alpha + s\beta \in \Psi$.

Let $f : X \to X$ be a group homomorphism such that

$$f(x_\alpha)x_\alpha^{-1} \in \langle X_\beta \mid \beta > \alpha\rangle \quad \textit{for all } x_\alpha \in X_\alpha, \textit{ all } \alpha \in \Psi_{\mathrm{red}}. \tag{1.2.6}$$

Fix an ordering $o\colon \Psi_{\mathrm{red}} \to \{1,2,\dots,|\Psi_{\mathrm{red}}|\}$ of the factors of $\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha$. For all $\alpha \in \Psi_{\mathrm{red}}$ and all $(x_\beta)_{\beta<\alpha} \in \prod_{\substack{\beta\in\Psi_{\mathrm{red}}\\ \beta<\alpha}} X_\beta$ there exists a unique element $z_\alpha \in X_\alpha$ and a unique group homomorphism $\tilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ factoring through $X_\alpha/X_{2\alpha}$ such that

$$f\Bigg(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\Bigg) = \prod_{\alpha\in\Psi_{\mathrm{red}}} \big(z_\alpha\tilde{z}_\alpha(x_\alpha)x_\alpha\big) \quad \textit{for all } x_\alpha \in X_\alpha, \textit{ all } \alpha \in \Psi_{\mathrm{red}}. \tag{1.2.7}$$

Proof. Notice that condition (iii) of Lemma 1.6 holds by Remark 1.4 (d). Therefore the statement follows from Lemma 1.8. □

Remark 1.10. We complement Proposition 1.9 (resp. Lemma 1.8) with a few observations. Retain the notations of Proposition 1.9.

(a) The homomorphism $f : X \to X$ is necessarily an automorphism.

(b) Given $(x_\beta)_{\beta<\alpha} \in \prod_{\substack{\beta\in\Psi_{\mathrm{red}}\\ \beta<\alpha}} X_\beta$ and an ordering $o\colon \Psi_{\mathrm{red}} \to \{1, 2, \dots, |\Psi_{\mathrm{red}}|\}$ of the factors of $\prod_{\alpha\in\Psi_{\mathrm{red}}} X_\alpha$, the homomorphism $\tilde{z}_\alpha\colon X_\alpha \to X_{2\alpha}$ only depends on the x_β with $\beta < \alpha$ *and* $o(\beta) > o(\alpha)$ (cf. (1.2.5)). Consequently, for an appropriate choice of the ordering o, the homomorphism $\tilde{z}_\alpha$ does *not* depend on $(x_\beta)_{\beta<\alpha}$.

1.3. Valuations on root group data

This section is dedicated the study of *valuations* on a root group datum. These play a crucial role in the construction of the apartment in the building of G. In the case where G is the group of F-points of a connected reductive group the existence of a valuation needs the full strength of Bruhat-Tits theory.

Definition 1.11. [BT72, (6.2.1)] Let G be a group and $\big(Z, (U_\alpha, M_\alpha)_{\alpha\in\Phi}\big)$ a root group datum on G. A family $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ of maps $\varphi_\alpha\colon U_\alpha \to \mathbb{R}\cup\{\infty\}$ ($\alpha \in \Phi$) is called a *valuation* on the root group datum $\big(Z, (U_\alpha, M_\alpha)_{\alpha\in\Phi}\big)$ if the following conditions are satisfied:

(V_0) For each $\alpha \in \Phi$ the image of φ_α contains at least three elements.

(V_1) For all $\alpha \in \Phi$ and $r \in \mathbb{R}\cup\{\infty\}$ the set $U_{\alpha,r} := \varphi_\alpha^{-1}\big([r, \infty]\big)$ is a subgroup of U_α and we have $U_{\alpha,\infty} = \{1\}$.

(V_2) For all $\alpha \in \Phi$ and $m \in M_\alpha$ the function $U_{-\alpha}^* \to \mathbb{R}$, $u \mapsto \varphi_{-\alpha}(u) - \varphi_\alpha(mum^{-1})$ is constant.

(V_3) Let $\alpha, \beta \in \Phi$ and $r, s \in \mathbb{R}$; if $\beta \notin -\mathbb{R}_{>0}\alpha$, then the group of commutators $[U_{\alpha,r}, U_{\beta,s}]$ is contained in the subgroup of G generated by the $U_{n\alpha+m\beta,nr+ms}$ for $n, m \in \mathbb{N}$ with $n\alpha + m\beta \in \Phi$.

(V_4) If $\alpha, 2\alpha \in \Phi$, then $\varphi_{2\alpha}$ is the restriction of $2\varphi_\alpha$ to $U_{2\alpha}$.

(V_5) Let $\alpha \in \Phi$, $u \in U_\alpha$, and $u', u'' \in U_{-\alpha}$. If $u'uu'' \in M_\alpha$, then we have $\varphi_{-\alpha}(u') = -\varphi_\alpha(u)$.

Given $\alpha \in \Phi$ and $r \in \mathbb{R}$, we put $U_{\alpha,r+} := \bigcup_{s>r} U_{\alpha,s} \subseteq U_{\alpha,r}$ and $U_{\alpha,r,\kappa} := U_{\alpha,r}/U_{\alpha,r+}$.

Remark 1.12. Let $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ be a valuation of $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$. We collect some properties of φ [BT72, (6.2.2)].

(a) The family of subgroups $\big\{U_{\alpha,r} \,\big|\, r \in \mathbb{R}\big\}$ of U_α is a basis of open neighborhoods of 1 for the structure of a topological group on U_α.

(b) The valuation φ_α can be recovered from $\big\{U_{\alpha,r} \,\big|\, r \in \mathbb{R}\big\}$ via

$$\varphi_\alpha(u) = \sup\big\{r \in \mathbb{R}\cup\{\infty\} \,\big|\, u \in U_{\alpha,r}\big\} \quad \text{for } u \in U_\alpha.$$

(c) Condition (V_5) is equivalent to

(V_5') Let $\alpha \in \Phi$, $u \in U_\alpha$ and $u', u'' \in U_{-\alpha}$ with $u'uu'' \in M_\alpha$. Then we have $\varphi_{-\alpha}(u'') = -\varphi_\alpha(u)$.

From (V_1) it follows that $\varphi_\alpha(u^{-1}) = \varphi_\alpha(u)$ for $u \in U_\alpha$, $\alpha \in \Phi$. Given $u \in U_\alpha$ and $u', u'' \in U_{-\alpha}$ with $u'uu'' \in M_\alpha$, we also have $u''^{-1}u^{-1}u'^{-1} = (u'uu'')^{-1} \in M_\alpha$. Using ($\mathrm{V}_5$) we compute

$$\varphi_{-\alpha}(u'') = \varphi_{-\alpha}(u''^{-1}) = -\varphi_\alpha(u^{-1}) = -\varphi_\alpha(u).$$

This shows (V_5'). A similar argument shows that (V_5') implies (V_5).

(d) If $\Delta \subseteq \Phi$ is a basis, then φ is uniquely determined by $(\varphi_\alpha)_{\alpha\in\Delta}$ [BT72, Cor. (6.2.8)].

Definition 1.13. A valuation $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ on the root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ is called *discrete* if the set of values $\Gamma_\alpha := \varphi_\alpha(U_\alpha^*) \subseteq \mathbb{R}$ is discrete.

Example 1.14. (a) We consider the general linear group $\mathrm{GL}_n(F)$ together with the root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ defined in Example 1.5, (a). Given $\alpha_{ij} \in \Phi$, we define

$$\varphi_{\alpha_{ij}}\big(E_n + \lambda E_{ij}\big) := \omega(\lambda), \quad \text{for } \lambda \in F.$$

Then (V_0) and (V_4) are trivially satisfied. Given $r \in \mathbb{R}$, let $k \in \mathbb{Z}$ be minimal with $k \geq r$. Then $U_{\alpha_{ij},r} = U_{\alpha_{ij},k} = \left\{E_n + \lambda E_{ij} \,\middle|\, \lambda \in \pi_F^k O_F\right\}$ is a group and we have $U_{\alpha_{ij},\infty} = \{1\}$, whence ($\mathrm{V}_1$). The computation (1.2.3) shows that (V_3) is satisfied. For the verification of (V_5) and (V_2) we may reduce to the case of 2×2-matrices. The requirement that

$$\begin{pmatrix} 1 & 0 \\ \lambda' & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda'' & 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda\lambda'' & \lambda \\ \lambda''+\lambda'+\lambda'\lambda\lambda'' & \lambda\lambda'+1 \end{pmatrix} \overset{!}{\in} \begin{pmatrix} 0 & F^\times \\ F^\times & 0 \end{pmatrix}$$

forces $\lambda' = \lambda'' = -\lambda^{-1}$. Hence $\omega(\lambda') = \omega(\lambda'') = -\omega(\lambda)$ which implies (V_5) (and (V_5')). Similarly we compute for $x, y, \lambda \in F^\times$:

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ x^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & x\lambda y^{-1} \\ 0 & 1 \end{pmatrix}.$$

As $\omega(\lambda) - \omega(x\lambda y^{-1}) = \omega(y) - \omega(x)$ does not depend on λ, we see that (V_2) holds. Therefore, φ is a valuation.

(b) [BT72, Ex. (6.1.3) b)] Let $\mathbf{G}$ be an F-split connected reductive group over F, and fix a maximal torus $\mathbf{T}$ in $\mathbf{G}$. Then we have $\mathbf{Z_G}(\mathbf{T}) = \mathbf{T}$. The root groups $\mathbf{U}_\alpha$ are isomorphic to the additive group $\mathbb{G}_\mathrm{a}$. In fact, we may choose isomorphisms $\chi_\alpha \colon \mathbb{G}_\mathrm{a} \to \mathbf{U}_\alpha$, for each $\alpha \in \Phi$, such that the following conditions are satisfied [BT84, 3.2.2]:

(CH_1) For each $\alpha \in \Phi$ the isomorphisms χ_α and $\chi_{-\alpha}$ are *associated*, i. e. there exists a (unique) F-morphism $\zeta_\alpha \colon \mathrm{SL}_2 \to \mathbf{G}$ such that

$$\chi_{\alpha,R}(u) = \zeta_{\alpha,R}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \chi_{-\alpha,R}(u) = \zeta_{\alpha,R}\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$$

for all $u \in \mathbb{G}_\mathrm{a}(R) = R$ and all F-algebras R.

(CH$_2$) For all $\alpha, \beta \in \Phi$ there exists $\varepsilon \in \{\pm 1\}$ such that

$$\chi_{s_{\alpha,\alpha^\vee}(\beta),R}(u) = \zeta_{\alpha,R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \chi_{\beta,R}(\varepsilon u) \cdot \zeta_{\alpha,R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

for all $u \in \mathbb{G}_a(R) = R$ and F-algebras R.

The datum $(\chi_\alpha)_{\alpha\in\Phi}$ is called a *Chevalley system* in $\mathbf{G}$ (relative to $\mathbf{T}$). Notice that for each $\alpha \in \Phi$ we have

$$t \cdot \chi_{\alpha,R}(u) \cdot t^{-1} = \chi_{\alpha,R}\big(\alpha(t) \cdot u\big) \tag{1.3.1}$$

for all $t \in T$, $u \in \mathbb{G}_a(R) = R$ and F-algebras R [Hum98, 26.3 Thm. (c)]. For all $\alpha, \beta \in \Phi$ with $\beta \neq -\alpha$, and $i, j \in \mathbb{N}$ with $i\alpha + j\beta \in \Phi$ there exists $C_{\alpha,\beta;i,j} \in \mathbb{Z}$ such that the commutator formula

$$\big[\chi_{\alpha,R}(u), \chi_{\beta,R}(v)\big] = \prod_{\substack{i,j\in\mathbb{N} \\ i\alpha+j\beta\in\Phi}} \chi_{i\alpha+j\beta,R}\big(C_{\alpha,\beta;i,j} \cdot u^i v^j\big) \tag{1.3.2}$$

holds for all $u, v \in \mathbb{G}_a(R) = R$ and F-algebras R [BT84, 3.2.3] (see also [Hum98, 32.5 Lemma] for a proof). Recall from Example 1.5, (b) that $\big(T, (U_\alpha)_{\alpha\in\Phi}\big)$ is a generating root group datum of type Φ. We can use the above Chevalley system to define a valuation $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ for this root group datum by defining

$$\varphi_\alpha \colon U_\alpha \longrightarrow \mathbb{R}, \quad \varphi_\alpha\big(\chi_{\alpha,F}(\lambda)\big) = \omega(\lambda), \qquad (\lambda \in F) \tag{1.3.3}$$

for each $\alpha \in \Phi$. Again, (V$_0$) and (V$_4$) are trivial, and (V$_1$) is also clear. Condition (V$_3$) is a direct consequence of the commutator formula (1.3.2). We prove (V$_2$): note that M_α (1.2.1) is of the form Tm_α, where $m_\alpha := \zeta_{\alpha,F}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Given $\lambda \in F$, we compute

$$\begin{aligned}\varphi_\alpha\big(m_\alpha \chi_{-\alpha,F}(\lambda) m_\alpha^{-1}\big) &= (\varphi_\alpha \circ \zeta_{\alpha,F})\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\ &= (\varphi_\alpha \circ \zeta_{\alpha,F})\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \varphi_\alpha\big(\chi_{\alpha,F}(\lambda)\big).\end{aligned}$$

Hence (V$_2$) holds for m_α. It remains to prove that the map $U_\alpha^* \longrightarrow \mathbb{R}$, $u \mapsto \varphi_\alpha(u) - \varphi_\alpha(tut^{-1})$ is constant for each fixed $t \in T$. Using (1.3.1) we compute

$$\begin{aligned}\varphi_\alpha\big(t\chi_{\alpha,F}(\lambda)t^{-1}\big) &= \varphi_\alpha\big(\chi_{\alpha,F}\big(\alpha(t)\cdot\lambda\big)\big) = \omega\big(\alpha(t)\cdot\lambda\big) \\ &= \omega(\lambda) + \omega\big(\alpha(t)\big) = \varphi_\alpha\big(\chi_{\alpha,F}(\lambda)\big) + \omega\big(\alpha(t)\big)\end{aligned} \tag{1.3.4}$$

for $\lambda \in F^\times$. Thus, (V$_2$) holds true. Let $\lambda, \lambda', \lambda'' \in F$, $\lambda \neq 0$, such that

$$\begin{aligned}\chi_{-\alpha,F}(\lambda')\chi_{\alpha,F}(\lambda)\chi_{-\alpha,F}(\lambda'') &= \zeta_{\alpha,F}\left(\begin{pmatrix} 1 & 0 \\ -\lambda' & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\lambda'' & 1 \end{pmatrix}\right) \\ &= \zeta_{\alpha,F}\begin{pmatrix} 1-\lambda\lambda'' & \lambda \\ \lambda'\lambda\lambda'' - \lambda'' - \lambda' & 1-\lambda\lambda' \end{pmatrix} \overset{!}{\in} M_\alpha.\end{aligned}$$

This means $\lambda' = \lambda'' = \lambda^{-1}$. Therefore,

$$\varphi_{-\alpha}\big(\chi_{-\alpha,F}(\lambda')\big) = \omega(\lambda') = -\omega(\lambda) = -\varphi_{\alpha}\big(\chi_{\alpha,F}(\lambda)\big),$$

whence (V5). Thus, φ is indeed a valuation.

Lemma 1.15. *[BT72, (6.2.5)] Let $\varphi = (\varphi_\alpha)_{\alpha\in\Phi}$ be a valuation on a root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$.*

(i) Let $\lambda\colon \Phi \to \mathbb{R}_{>0}$ be a function that is constant on the irreducible components of Φ, and let $v \in V$. Define a family $\psi = (\psi_\alpha)_{\alpha\in\Phi}$ of maps $\psi_\alpha\colon U_\alpha \to \mathbb{R}\cup\{\infty\}$ by

$$\psi_\alpha(u) = \lambda(\alpha)\cdot\varphi_\alpha(u) + \langle\alpha, v\rangle \quad \textit{for } \alpha\in\Phi \textit{ and } u \in U_\alpha.$$

Then ψ is also a valuation, denoted by $\lambda\varphi + v$ (or by $\varphi + v$ if $\lambda(\alpha) = 1$ for all $\alpha \in \Phi$). This defines a free action of V on the set of all valuations of $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$.

(ii) Recall the map ${}^{\mathrm{v}}\nu\colon N \to W_0$ (1.2.2). Let $n \in N$ and put $w := {}^{\mathrm{v}}\nu(n) \in W_0$. Define a family $\psi = (\psi_\alpha)_{\alpha\in\Phi}$ of maps $\psi_\alpha\colon U_\alpha \to \mathbb{R}\cup\{\infty\}$ by

$$\psi_\alpha(u) = \varphi_{w^{-1}(\alpha)}(n^{-1}un) \quad \textit{for } \alpha\in\Phi \textit{ and } u \in U_\alpha.$$

Then $\psi = (\psi_\alpha)_{\alpha\in\Phi}$ is again a valuation, denoted by $n.\varphi$. In this way we obtain an action of N on the set of all valuations of $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$.

(iii) Given $n \in N$, $v \in V$, and $\lambda\colon \Phi \to \mathbb{R}_{>0}$ as in (i), we have

$$n.(\lambda\varphi + v) = \lambda\cdot(n.\varphi) + {}^{\mathrm{v}}\nu(n)(v).$$

Proof. (i) Clearly, the image of ψ_α contains at least three elements, whence (V_0) holds. For each $r \in \mathbb{R}\cup\{\infty\}$ and $\alpha\in\Phi$ the set

$${}^{\psi}U_{\alpha,r} := \psi_\alpha^{-1}\big([r,\infty]\big) = \varphi_\alpha^{-1}\big([\lambda(\alpha)^{-1}\cdot(r - \langle\alpha,v\rangle),\infty]\big) = U_{\alpha,\lambda(\alpha)^{-1}(r-\langle\alpha,v\rangle)}$$

is a group, and we have $\psi_\alpha^{-1}(\infty) = \{1\}$. This shows ($V_1$). Let $\alpha\in\Phi$ and $m \in M_\alpha$. For each $u \in U_{-\alpha}^*$ we compute, using $\lambda(-\alpha) = \lambda(\alpha)$,

$$\begin{aligned}\psi_{-\alpha}(u) - \psi_\alpha(mum^{-1}) &= \lambda(-\alpha)\varphi_{-\alpha}(u) + \langle-\alpha, v\rangle \\ &\qquad - \lambda(\alpha)\varphi_\alpha(mum^{-1}) - \langle\alpha, v\rangle \\ &= \lambda(\alpha)\cdot\big(\varphi_{-\alpha}(u) - \varphi_\alpha(mum^{-1})\big) - 2\langle\alpha, v\rangle.\end{aligned}$$

Hence condition (V_2) for φ implies (V_2) for ψ. If $\alpha, 2\alpha \in \Phi$, we have, using $\lambda(\alpha) = \lambda(2\alpha)$,

$$\psi_{2\alpha}(u) = \lambda(2\alpha)\varphi_{2\alpha}(u) + \langle 2\alpha, v\rangle = 2\lambda(\alpha)\varphi_\alpha(u) + 2\langle\alpha, v\rangle = 2\psi_\alpha(u)$$

for each $u \in U_{2\alpha}$. Therefore, (V_4) holds. Similarly, given $\alpha\in\Phi$, $u \in U_\alpha$, and $u', u'' \in U_{-\alpha}$ with $u'uu'' \in M_\alpha$, we compute

$$\psi_{-\alpha}(u') = \lambda(-\alpha)\varphi_{-\alpha}(u') + \langle-\alpha, v\rangle = -\lambda(\alpha)\varphi_\alpha(u) - \langle\alpha, v\rangle = -\psi_\alpha(u),$$

whence (V5). It remains to show (V3). Let $\alpha, \beta \in \Phi$ with $\beta \notin -\mathbb{R}_{>0}\alpha$. If α and β belong to different irreducible components of Φ then $n\alpha+m\beta \notin \Phi$ for all $n, m \in \mathbb{N}$, and (V3) follows immediately from (V3) for φ. Hence, we may assume that α and β belong to the same irreducible component, say Ψ, of Φ. Given $n, m \in \mathbb{N}$ with $n\alpha + m\beta \in \Phi$, we deduce $n\alpha + m\beta \in \Psi$ and $\lambda(\alpha) = \lambda(\beta) = \lambda(n\alpha + m\beta)$. Thus, we compute

$$\begin{aligned} n \cdot \big(\lambda(\alpha)r + \langle\alpha, v\rangle\big) &+ m\cdot\big(\lambda(\beta)s + \langle\beta, v\rangle\big) \\ &= \lambda(n\alpha + m\beta)\cdot(nr + ms) + \langle n\alpha + m\beta, v\rangle. \end{aligned}$$

Applying (V3) for φ, and noticing ${}^{\psi}U_{\alpha,\lambda(\alpha)r+\langle\alpha,v\rangle} = U_{\alpha,r}$, it follows that the commutator $[{}^{\psi}U_{\alpha,\lambda(\alpha)r+\langle\alpha,v\rangle}, {}^{\psi}U_{\beta,\lambda(\beta)s+\langle\beta,v\rangle}] = [U_{\alpha,r}, U_{\beta,s}]$ is contained in the subgroup generated by all

$$\begin{aligned} U_{n\alpha+m\beta,nr+ms} &= {}^{\psi}U_{n\alpha+m\beta,\lambda(n\alpha+m\beta)(nr+ms)+\langle n\alpha+m\beta,v\rangle} \\ &= {}^{\psi}U_{n\alpha+m\beta,n\cdot(\lambda(\alpha)r+\langle\alpha,v\rangle)+m\cdot(\lambda(\beta)s+\langle\beta,v\rangle)}, \end{aligned}$$

where $n, m \in \mathbb{N}$ with $n\alpha + m\beta \in \Phi$. This proves (V3) and hence that ψ is a valuation. Since Φ generates V^*, it follows that the action of V on the set of all valuations is free.

(ii) Notice that by Remark 1.4, (d) we have $n^{-1}U_\alpha n = U_{w^{-1}(\alpha)}$, which shows that ψ_α is well-defined for $\alpha \in \Phi$, and that (V0) holds. For each $r \in \mathbb{R} \cup \{\infty\}$ we have ${}^{\psi}U_{\alpha,r} = \psi_\alpha^{-1}([r, \infty]) = nU_{w^{-1}(\alpha),r}n^{-1}$, whence (V1) and (V3). Let $\alpha \in \Phi$ and $m \in M_\alpha$. From Remark 1.4, (b) it follows that $n^{-1}M_\alpha n = M_{w^{-1}(\alpha)}$. Since (V2) holds for φ, we conclude that

$$\begin{aligned} &\psi_{-\alpha}(u)-\psi_\alpha(mum^{-1}) \\ &\qquad = \varphi_{-w^{-1}(\alpha)}(n^{-1}un) - \varphi_{w^{-1}(\alpha)}\big((n^{-1}mn)(n^{-1}un)(n^{-1}mn)^{-1}\big) \end{aligned}$$

does not depend on $u \in U_{-\alpha}^*$. This shows (V2) for ψ. Finally, (V4) and (V5) are immediate. Hence, $\psi = (\psi_\alpha)_{\alpha\in\Phi}$ is a valuation.

(iii) Write again $w := {}^{\mathrm{v}}\nu(n)$. A direct computation yields

$$\begin{aligned} [n.(\lambda\varphi + v)]_\alpha(u) &= (\lambda\varphi + v)_{w^{-1}(\alpha)}(n^{-1}un) \\ &= \lambda\big(w^{-1}(\alpha)\big)\cdot\varphi_{w^{-1}(\alpha)}(n^{-1}un) + \langle w^{-1}(\alpha), v\rangle \\ &= [\lambda\cdot(n.\varphi) + w(v)]_\alpha(u) \end{aligned}$$

for all $\alpha \in \Phi$ and $u \in U_\alpha$.

□

The following proposition shows that the action of N on the set of valuations can be made explicit and also behaves nicely in a sense that will become apparent in the next section.

Proposition 1.16. *Let φ be a valuation on the root group datum $\big(Z,(U_\alpha,M_\alpha)_{\alpha\in\Phi}\big)$. Given $\alpha\in\Phi$ and $r\in\Gamma_\alpha=\varphi_\alpha(U_\alpha^*)$, we put (cf. Remark 1.4, (b))*

$$M_{\alpha,r} := M_\alpha \cap U_{-\alpha}\varphi_\alpha^{-1}(\{r\})U_{-\alpha} = m\big(U_{\alpha,r}\setminus U_{\alpha,r+}\big). \tag{1.3.5}$$

For $i=1,\dots,k$ let $\alpha_i\in\Phi$, $r_i\in\Gamma_{\alpha_i}$ and $m_i\in M_{\alpha_i,r_i}$. Put $n:=m_1\cdots m_k$ and $s_i:=s_{\alpha_i,\alpha_i^\vee}$. Then we have $n.\varphi=\varphi-v$ with

$$v=\sum_{i=1}^k r_i\cdot(s_1\circ\cdots\circ s_{i-1})(\alpha_i^\vee)\in V.$$

Proof. See [BT72, Prop. (6.2.7)]. □

Lemma 1.17. *Let φ be a valuation on the root group datum $\big(Z,(U_\alpha,M_\alpha)_{\alpha\in\Phi}\big)$. For each $z\in Z$ there exists a unique vector $\nu(z)\in V$ with $z.\varphi=\varphi+\nu(z)$. In this way we obtain a group homomorphism*

$$\nu\colon Z\longrightarrow V. \tag{1.3.6}$$

Proof. See [BT72, Prop. (6.2.10), Proof of (i)]. Fix a basis Δ of Φ, and let $z\in Z$ and $\alpha\in\Delta$. Given $m\in M_\alpha$, we also have $zm\in M_\alpha$. By (V2) there exist constants $c(\alpha,m),c(\alpha,zm)\in\mathbb{R}$ with $\varphi_\alpha(mum^{-1})=\varphi_{-\alpha}(u)+c(\alpha,m)$ and $\varphi_{-\alpha}(u)=\varphi_\alpha\big((zm)u(zm)^{-1}\big)+c(\alpha,zm)$ for each $u\in U_{-\alpha}^*$. Given $u\in U_\alpha^*$, we compute

$$\begin{aligned}(z.\varphi)_\alpha(u)=\varphi_\alpha(z^{-1}uz)&=\varphi_\alpha\big(m(zm)^{-1}u(zm)m^{-1}\big)\\&=\varphi_{-\alpha}\big((zm)^{-1}u(zm)\big)+c(\alpha,m)\\&=\varphi_\alpha(u)+c(\alpha,zm)+c(\alpha,m).\end{aligned}$$

Since Δ is also a basis of V^*, there exists exactly one $\nu(z)\in V$ with $\langle\alpha,\nu(z)\rangle=c(\alpha,zm)+c(\alpha,m)$ for all $\alpha\in\Delta$. Then $\varphi+\nu(z)$ is a new valuation with $(z.\varphi)_\alpha=\big(\varphi+\nu(z)\big)_\alpha$ for all $\alpha\in\Delta$. By Remark 1.12, (d) we obtain $z.\varphi=\varphi+\nu(z)$ as desired. Since $Z\subseteq\operatorname{Ker}({}^v\nu\colon N\to W_0)$ it follows from Lemma 1.15, (iii) that the map $\nu\colon Z\to V$ thus defined is a group homomorphism. □

Definition 1.18. The valuation φ is called *special* if $0\in\Gamma_\alpha=\varphi_\alpha(U_\alpha^*)$ for all $\alpha\in\Phi_{\mathrm{red}}$.

Proposition 1.19. *Let φ be a valuation of the root group datum $\big(Z,(U_\alpha)_{\alpha\in\Phi}\big)$. Then there exists $v\in V$ such that $\varphi+v$ is special.*

Proof. By [BT72, Cor. (6.2.15)] there exists a valuation φ with $0\in\Gamma_\alpha$ for all $\alpha\in\{\beta\in\Phi\mid 2\beta\notin\Phi\}$. Since $(\frac{1}{2}\Gamma_{2\alpha})\subseteq\Gamma_\alpha$ by (V4), it follows that φ is special. □

We now fix a connected reductive group $\mathbf{G}$ and a maximal F-split torus $\mathbf{T}$. Assume there exists a valuation φ on the root group datum $\big(Z,(U_\alpha)_{\alpha\in\Phi}\big)$ of G (see Example 1.5, (b)). Recall the vector space $V=\big(X_*(T)/X_*(C)\big)\otimes_{\mathbb{Z}}\mathbb{R}$ (1.1.2). Notice that the restriction map from $\mathbf{Z}_{\mathbf{G}}(\mathbf{T})$ to $\mathbf{T}$ embeds $X^*(Z)$ into $X^*(T)$ as a subgroup of finite index [Ren10, V.2.6.

Lemme, Démonstration]. Hence, given $\chi \in X^*(T)$, there exists $n \in \mathbb{N}$ such that $n\chi \in X^*(Z)$, and we may define

$$(\omega \circ \chi)(z) := \frac{1}{n} \cdot \omega\big((n\chi)(z)\big) \in \mathbb{R} \quad \text{for all } z \in Z.$$

Clearly, this definition is independent of n.

Definition 1.20. The valuation φ on the root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ of G is called *compatible with* ω if we have

$$\langle \alpha, \nu(z)\rangle = -(\omega \circ \alpha)(z) \quad \text{for all } z \in Z, \text{ all } \alpha \in \Phi.$$

In view of Lemma 1.17 this means that φ is compatible with ω if

$$\varphi_\alpha(zuz^{-1}) = \varphi_\alpha(u) + (\omega \circ \alpha)(z) \quad \text{for all } z \in Z,\, u \in U_\alpha, \text{ and } \alpha \in \Phi.$$

We remark that with φ also all valuations of the form $\varphi + v$, for $v \in V$, are compatible with ω.

Example 1.21. If $\mathbf{G}$ is an F-split connected reductive group over F and $\mathbf{T}$ is a maximal torus in $\mathbf{G}$, then the computation in (1.3.4) shows that the valuation φ on the root group datum $\big(T, (U_\alpha)_{\alpha\in\Phi}\big)$ constructed in Example 1.14, (b) is compatible with ω.

Theorem 1.22. *Let $\mathbf{G}$ be a connected reductive group over F and let $\mathbf{T}$ be a maximal F-split torus. Let $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ be the associated generating root group datum of type $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ (Example 1.5, (b)). Then there exists a valuation on $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ that is discrete, special, and compatible with ω.*

Proof. The existence of a discrete valuation φ that is compatible with ω is part of the statement of [BT84, 5.1.20. Thm. and 5.1.23. Prop.]; see also [BT84, 5.1.15] for the statement that $A_\natural$ is non-empty. Now, Proposition 1.19 shows that there exists $v \in V$ such that the discrete valuation $\varphi + v$, which is again compatible with ω, is special. $\square$

Remark 1.23. Assume the situation of Theorem 1.22. By [Vig16, (37)] the subgroups $U_{\alpha,r}$, for $\alpha \in \Phi$ and $r \in \Gamma_\alpha$, are compact open in U_α. Hence $\big\{U_{\alpha,r} \,\big|\, r \in \Gamma_\alpha\big\}$ is a basis of compact open neighborhoods of the neutral element in U_α, $\alpha \in \Phi$.

1.4. The apartment of a reductive group and affine roots

We fix a connected reductive group $\mathbf{G}$ over F, a maximal F-split torus $\mathbf{T}$, and a discrete special valuation φ_0 on the associated generating root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ of type $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ which is compatible with ω (Theorem 1.22). Recall also the vector space $V = \big(X_*(T)/X_*(C)\big) \otimes_{\mathbb{Z}} \mathbb{R}$ (1.1.2). We now come to the central definition:

Definition 1.24. The affine space $\mathscr{A}$ under V, defined by

$$\mathscr{A} := \{\varphi_0 + v \mid v \in V\}, \tag{1.4.1}$$

is called the *apartment* of G (cf. Lemma 1.15). By [BT84, 5.1.23] the apartment consists of all discrete valuations on $\big(Z, (U_\alpha)_{\alpha\in\Phi}\big)$ that are compatible with ω. The fixed W_0-invariant scalar product on V endows $\mathscr{A}$ with a Euclidean metric.

Recall the notation $\Gamma_\alpha := \varphi_{0,\alpha}(U_\alpha^*) \subseteq \mathbb{R}$ for $\alpha \in \Phi$. For $\alpha \in \Phi$ we put

$$\Gamma'_\alpha := \left\{\varphi_{0,\alpha}(u) \,\middle|\, u \in U_\alpha^* \text{ and } \varphi_{0,\alpha}(u) = \sup \varphi_{0,\alpha}(uU_{2\alpha})\right\}. \tag{1.4.2}$$

Notice that $\Gamma'_\alpha = \Gamma_\alpha$ whenever $2\alpha \notin \Phi$ (in which case we have put $U_{2\alpha} := \{1\}$). In our context Γ'_α is never empty (as φ_0 is discrete) [BT84, 4.2.21].

Lemma 1.25. *(i) With the notations from above we have $\Gamma_\alpha = \Gamma'_\alpha \cup (\frac{1}{2}\Gamma_{2\alpha})$ for all $\alpha \in \Phi$.*

(ii) For each $r \in \Gamma_\alpha \smallsetminus \Gamma'_\alpha$ the inclusion $U_{2\alpha,2r} \hookrightarrow U_{\alpha,r}$ induces an isomorphism

$$U_{2\alpha,2r}/U_{2\alpha,2r+} \cong U_{\alpha,r}/U_{\alpha,r+}.$$

Proof. (i) See [BT72, (6.2.2)]. Fix $\alpha \in \Phi$. It is clear that $\Gamma'_\alpha \cup (\frac{1}{2}\Gamma_{2\alpha}) \subseteq \Gamma_\alpha$ in view of (V4). Conversely, let $u \in U_\alpha^*$. We may assume $\varphi_{0,\alpha}(u) \notin \Gamma'_\alpha$. Then there exists $v \in U_{2\alpha}^*$ with $\varphi_{0,\alpha}(uv) > \varphi_{0,\alpha}(u)$. As $U_{\alpha,r}$ is a group for all $r \in \mathbb{R}$, this implies

$$\varphi_{0,\alpha}(u) = \varphi_{0,\alpha}(u^{-1}) = \varphi_{0,\alpha}(u^{-1}uv) = \varphi_{0,\alpha}(v) = \tfrac{1}{2} \cdot \varphi_{0,2\alpha}(v) \in \tfrac{1}{2}\Gamma_{2\alpha}.$$

This establishes $\Gamma_\alpha \subseteq \Gamma'_\alpha \cup (\frac{1}{2}\Gamma_{2\alpha})$ and hence equality.

(ii) See [Vig16, Lem 3.8]. Let $r \in \Gamma_\alpha \smallsetminus \Gamma'_\alpha$ and take $u \in U_\alpha^*$ with $r = \varphi_{0,\alpha}(u)$. Then there exists $v \in U_{2\alpha}^*$ with $\varphi_{0,\alpha}(v^{-1}u) = \varphi_{0,\alpha}(u^{-1}v) > \varphi_{0,\alpha}(u^{-1}) = \varphi_{0,\alpha}(u)$; it necessarily satisfies $\varphi_{0,\alpha}(v) \geq r$. Hence, $u = v \cdot (v^{-1}u) \in U_{2\alpha,2r}U_{\alpha,r+}$. This shows that the group homomorphism $U_{2\alpha,2r} \to U_{\alpha,r}/U_{\alpha,r+}$ is surjective. It is straightforward to see $U_{2\alpha,2r+} = U_{2\alpha,2r} \cap U_{\alpha,r+}$ using $\frac{1}{2}\Gamma_{2\alpha} \subseteq \Gamma_\alpha$. From this the assertion follows. □

Definition 1.26. For $\alpha \in V^*$ and $r \in \mathbb{R}$ we denote by $a_{\alpha,r}$ the closed half-space

$$a_{\alpha,r} := \{x \in \mathcal{A} \mid \langle \alpha, x - \varphi_0 \rangle + r \geq 0\},$$

and by

$$H_{\alpha,r} := \partial a_{\alpha,r} := \{x \in \mathcal{A} \mid \langle \alpha, x - \varphi_0 \rangle + r = 0\}$$

the *wall* of $a_{\alpha,r}$.

The closed half-spaces of the form $a_{\alpha,r}$ with $\alpha \in \Phi$ and $r \in \Gamma'_\alpha$ are called the *affine roots* of $\mathcal{A}$. Denote by Φ^{aff} the set of affine roots of $\mathcal{A}$ and by

$$\mathfrak{H} := \left\{H_{\alpha,r} \,\middle|\, \alpha \in \Phi, r \in \Gamma'_\alpha\right\} = \left\{H_{\alpha,r} \,\middle|\, \alpha \in \Phi_{\mathrm{red}}, r \in \Gamma_\alpha\right\} \tag{1.4.3}$$

the set of hyperplanes in $\mathcal{A}$.

Proposition 1.27. *Consider the apartment $\mathcal{A}$ of G together with its set of hyperplanes $\mathfrak{H}$.*

(i) The group N acts on the Euclidean affine space $\mathcal{A}$ via a group homomorphism $\nu : N \to$ $\operatorname{Aut}\mathcal{A}$ given by $\nu(n) \colon \varphi \mapsto n.\varphi$, for $n \in N$, such that $\nu(Z) \subseteq V$ and $\nu|_Z \colon Z \to V$ coincides with (1.3.6). *Under the canonical morphism* $\operatorname{Aut}\mathcal{A} \to \operatorname{Aut} V$, $f \mapsto {}^{\mathrm{v}}f$ *the map ${}^{\mathrm{v}}(\,\cdot\,) \circ \nu$ coincides with ${}^{\mathrm{v}}\nu$* (1.2.2).

(ii) Let $\alpha \in \Phi$ and $r \in \Gamma_\alpha$. Let $m \in M_{\alpha,r}$ (1.3.5). *Then $\nu(m)$ is the orthogonal reflection, denoted by $s_{\alpha,r}$ or $s_{H_{\alpha,r}}$, through the hyperplane $H_{\alpha,r}$.*

(iii) *Let* $n \in N$*, put* $w := {}^{\mathrm{v}}\nu(n)$*, and let* $a = a_{\alpha,r} \in \Phi^{\mathrm{aff}}$*. Then* $\nu(n)(a) = a_{w(\alpha),r-\langle w(\alpha),n.\varphi_0-\varphi_0\rangle} \in \Phi^{\mathrm{aff}}$ *and*

$$nU_{\alpha,r}n^{-1} = U_{w(\alpha),r-\langle w(\alpha),n.\varphi_0-\varphi_0\rangle}.$$

(iv) *The action of* N *on* $\mathcal{A}$ *induces an action on* $\mathfrak{H}$.

Proof. See [BT72, Prop. (6.2.10)]. Let $n \in N$. We first prove that $\nu(n)$ acts on $\mathcal{A}$ as a Euclidean affine homomorphism. Let N' be the subgroup of N generated by the $M_{\alpha,r}$ for $\alpha \in \Phi$ and $r \in \Gamma_\alpha$. If $n \in N'$, then Proposition 1.16 shows $\nu(n)(\varphi_0) = n.\varphi_0 \in \mathcal{A}$. If $n \in Z$, then Lemma 1.17 shows $\nu(n)(\varphi_0) = \varphi_0 + \nu(n) \in \mathcal{A}$ (we view $V \subseteq \operatorname{Aut}\mathcal{A}$ as the subgroup of translations, which justifies the notation). Since N is generated by Z and N' this shows that N acts on $\mathcal{A}$. By Lemma 1.15, (iii) we have $\nu(n)(\varphi_0 + v) = \nu(n)(\varphi_0) + {}^{\mathrm{v}}\nu(n)(v) \in \mathcal{A}$ for $v \in V$. As ${}^{\mathrm{v}}\nu(n)\colon V \to V$ is an orthogonal linear map (with respect to the fixed W_0-invariant scalar product), this shows that $\nu(n)$ acts on $\mathcal{A}$ as a Euclidean affine homomorphism. This establishes (i).

Let $m \in M_{\alpha,r}$ and $v \in V$. Then ${}^{\mathrm{v}}\nu(m) = s_{\alpha,\alpha^\vee}$ and hence, using Proposition 1.16,

$$\begin{aligned}\nu(m)(\varphi_0 + v) = m.\varphi_0 + s_{\alpha,\alpha^\vee}(v) &= \varphi_0 - r\alpha^\vee + v - \langle \alpha, v\rangle \cdot \alpha^\vee \\ &= \varphi_0 + v - \big(\langle \alpha, v\rangle + r\big) \cdot \alpha^\vee.\end{aligned}$$

Thus, $\nu(m)(x) = x$ if and only if $x \in H_{\alpha,r} = \{x \in \mathcal{A} \mid \langle \alpha, x - \varphi_0\rangle + r = 0\}$. Hence, $\nu(m)$ is the orthogonal reflection on $\mathcal{A}$ through $H_{\alpha,r}$, proving (ii).

Let $n \in N$ and $a = a_{\alpha,r} \in \Phi^{\mathrm{aff}}$. Let $u \in U_\alpha$ and write $w := {}^{\mathrm{v}}\nu(n)$. We compute

$$\begin{aligned}\varphi_{0,w(\alpha)}(nun^{-1}) = (n^{-1}.\varphi_0)_\alpha(u) &= \varphi_{0,\alpha}(u) - \langle \alpha, \varphi_0 - n^{-1}.\varphi_0\rangle \\ &= \varphi_{0,\alpha}(u) - \langle w(\alpha), n.\varphi_0 - \varphi_0\rangle.\end{aligned}$$

It follows that $nU_{\alpha,r}n^{-1} = U_{w(\alpha),r-\langle w(\alpha),n.\varphi_0-\varphi_0\rangle}$. Finally, we have

$$\begin{aligned}\nu(n)(a_{\alpha,r}) &= \nu(n)\big(\{x \in \mathcal{A} \mid \langle \alpha, x - \varphi_0\rangle + r \geq 0\}\big) \\ &= \big\{y \in \mathcal{A} \,\big|\, \langle \alpha, \nu(n^{-1})(y) - \varphi_0\rangle + r \geq 0\big\} \\ &= \big\{y \in \mathcal{A} \,\big|\, \langle w(\alpha), y - n.\varphi_0\rangle + r \geq 0\big\} \\ &= \big\{y \in \mathcal{A} \,\big|\, \langle w(\alpha), y - \varphi_0\rangle + r - \langle w(\alpha), n.\varphi_0 - \varphi_0\rangle \geq 0\big\} \\ &= a_{w(\alpha),r-\langle w(\alpha),n.\varphi_0-\varphi_0\rangle} \in \Phi^{\mathrm{aff}}.\end{aligned}$$

This shows (iii). Since $H_{\alpha,r} = \partial a_{\alpha,r}$ for $\alpha \in \Phi$, $r \in \Gamma'_\alpha$, the computation in (iii) shows (iv). □

We end this section by defining the finer structure on the apartment $\mathcal{A}$.

Definition 1.28. Two points $x, y \in \mathcal{A}$ are called *equivalent* if $x \in a \iff y \in a$ holds for all affine roots $a \in \Phi^{\mathrm{aff}}$. In other words, x and y are equivalent if for each hyperplane $H \in \mathfrak{H}$ we have either $x, y \in H$, or x and y are contained in the same connected component of $\mathcal{A} \setminus H$.

The equivalence classes are called *faces*. They are open in the affine subspace of $\mathcal{A}$ that they generate. Thus it makes sense to speak of the *dimension* of a face. The faces of maximal dimension are called *chambers* or *alcoves*; these are exactly the connected components of $\mathcal{A} \setminus \bigcup_{H\in\mathfrak{H}} H$.

The set $\mathcal{F}$ of faces of $\mathcal{A}$ becomes a partially ordered set by letting $\mathfrak{F} < \mathfrak{F}'$ if $\mathfrak{F}$ is contained in the topological closure $\overline{\mathfrak{F}'}$ of $\mathfrak{F}'$. Given two faces $\mathfrak{F}, \mathfrak{F}' \in \mathcal{F}$ with $\mathfrak{F} < \mathfrak{F}'$, we say that $\mathfrak{F}$ is a *face* of $\mathfrak{F}'$; if moreover $\dim \mathfrak{F}' = \dim \mathfrak{F} + 1$, we call $\mathfrak{F}$ a *facet* of $\mathfrak{F}'$. The action of N on $\mathcal{A}$ induces an order preserving action on $\mathcal{F}$.

A *wall* of an alcove $\mathfrak{C}$ is an affine hyperplane generated by a facet of $\mathfrak{C}$. We denote by $\mathfrak{H}(\mathfrak{C})$ the set of walls of $\mathfrak{C}$. Given any other alcove $\mathfrak{C}'$, we say that a hyperplane $H \in \mathfrak{H}$ *separates* $\mathfrak{C}$ and $\mathfrak{C}'$ if for some functional $\xi \in V^*$ with $\operatorname{Ker}\xi = H - \varphi_0$ we have $\langle \xi, \mathfrak{C} - \varphi_0\rangle > 0$ and $\langle \xi, \mathfrak{C}' - \varphi_0\rangle < 0$.

We fix, once and for all, an alcove $\mathfrak{C}$ in $\mathcal{A}$ with $\varphi_0 \in \overline{\mathfrak{C}}$. We call $\mathfrak{C}$ the *fundamental alcove*. We let Δ be the unique basis of Φ contained in $\Phi^+ := \{\alpha \in \Phi \mid \langle \alpha, \mathfrak{C} - \varphi_0\rangle \geq 0\}$ (see Remark 1.2 (c)). Then $H_{s_{\alpha,\alpha^\vee}} = \varphi_0 + \operatorname{Ker}\alpha$ is a wall of $\mathfrak{C}$ for all $\alpha \in \Delta$.

1.5. The affine Weyl group

We keep the notations of section 1.4.

Given $H \in \mathfrak{H}$, we denote by s_H the orthogonal reflection through H. Put $S(\mathfrak{H}) := \{s_H \mid H \in \mathfrak{H}\}$. Given $s \in S(\mathfrak{H})$, we denote by $H_s := \{x \in \mathcal{A} \mid s(x) = x\} \in \mathfrak{H}$ the corresponding hyperplane.

Definition 1.29. Put $\widetilde{W} := \nu(N) \subseteq \operatorname{Aut}\mathcal{A}$. Let W^{aff} be the subgroup of $\widetilde{W}$ generated by $S(\mathfrak{H})$. Notice that $w s_H w^{-1} = s_{w(H)}$ and $wH_s = H_{wsw^{-1}}$ for all $w \in \widetilde{W}$, $H \in \mathfrak{H}$, and $s \in S(\mathfrak{H})$. Since $\widetilde{W}$ acts on $\mathfrak{H}$ by Proposition 1.27 (iv), it follows that W^{aff} is a normal subgroup of $\widetilde{W}$. We call W^{aff} the *affine Weyl group* of G.

Proposition 1.30. *The stabilizer W_{φ_0} of φ_0 in $\widetilde{W}$ identifies with the finite Weyl group W_0 under the canonical map* $\operatorname{Aut}\mathcal{A} \to \operatorname{Aut} V$, $f \mapsto {}^{\mathrm{v}}f$.

We have the semidirect products $\widetilde{W} = W_0 \ltimes (\widetilde{W} \cap V)$ and $W^{\mathrm{aff}} = W_0 \ltimes (W^{\mathrm{aff}} \cap V)$. Moreover, $W^{\mathrm{aff}} \cap V$ is generated by the translations $t_{r\alpha^\vee}$ by $r\alpha^\vee$ for $\alpha \in \Phi_{\mathrm{red}}$ and $r \in \Gamma_\alpha$.

Proof. See [BT72, Prop. (6.2.19)]. It is clear that the restriction of ${}^{\mathrm{v}}(\cdot)$ to W_{φ_0} is injective. Recall that φ_0 is a special valuation, i. e. we have $0 \in \Gamma_\alpha$ for all $\alpha \in \Phi_{\mathrm{red}}$. Given $\alpha \in \Phi_{\mathrm{red}}$, the orthogonal reflection $s_{\alpha,0}$ through $H_{\alpha,0}$ lies in W_{φ_0} (since $\varphi_0 \in H_{\alpha,0}$). Therefore, W_{φ_0} surjects onto W_0.

It is a standard exercise to show that V (considered as the subgroup of translations) is a normal subgroup of $\operatorname{Aut}\mathcal{A}$, and $\operatorname{Aut}\mathcal{A} = \operatorname{Aut} V \ltimes V$. We thus deduce $\widetilde{W} = W_{\varphi_0} \ltimes (\widetilde{W} \cap V)$ and $W^{\mathrm{aff}} = W_{\varphi_0} \ltimes (W^{\mathrm{aff}} \cap V)$.

For any $\alpha \in \Phi_{\mathrm{red}}$ and $r \in \Gamma_\alpha$ we have $s_{\alpha,0}, s_{\alpha,r} \in W^{\mathrm{aff}}$ and hence $t_{r\alpha^\vee} = s_{\alpha,0}s_{\alpha,r} \in W^{\mathrm{aff}} \cap V$. Let W' be the subgroup of $W^{\mathrm{aff}} \cap V$ generated by the translations $t_{r\alpha^\vee}$ for

$\alpha \in \Phi_{\mathrm{red}}$ and $r \in \Gamma_\alpha$. Then W' is normalized by W_0, since $s_{\beta,0} t_{r\alpha^\vee} s_{\beta,0}^{-1} = t_{r s_{\beta,\beta^\vee}(\alpha^\vee)}$. Since also $s_{\alpha,r} = s_{\alpha,0} t_{r\alpha^\vee}$ for all $\alpha \in \Phi_{\mathrm{red}}$ and $r \in \Gamma_\alpha$, this shows that W' and W_0 generate W^{aff}. We conclude $W' = W^{\mathrm{aff}} \cap V$. □

Proposition 1.31. *Recall the basis Δ of Φ and that φ_0 is discrete and special. We have*

$$W^{\mathrm{aff}} \cap V = \left\{ \sum_{\alpha \in \Delta} r_\alpha \alpha^\vee \,\middle|\, r_\alpha \in \langle \Gamma_\alpha \rangle \text{ for } \alpha \in \Delta \right\} \quad \textit{and}$$
$$\widetilde{W} \cap V \subseteq \left\{ \sum_{\alpha \in \Delta} r_\alpha \varpi_\alpha \,\middle|\, r_\alpha \in \langle \Gamma_\alpha \rangle \text{ for } \alpha \in \Delta \right\},$$

where $\{\varpi_\alpha \mid \alpha \in \Delta\}$ is the dual basis of Δ in V. In particular, $W^{\mathrm{aff}} \cap V$ is a lattice in V of rank $\dim V = |\Delta|$.

Proof. See [BT72, Prop. (6.2.20)]. We remark that $\langle \Gamma_\alpha \rangle$, $\alpha \in \Delta$, is a discrete subgroup of $\mathbb{R}$ and hence equal to $\varepsilon_\alpha \mathbb{Z}$ for some $\varepsilon_\alpha > 0$. This follows from $0 \in \Gamma_\alpha$ and $\Gamma'_\alpha = \Gamma'_\alpha + 2\langle \Gamma_\alpha \rangle$ [BT72, Cor. (6.2.16)]. □

Proposition 1.32. *There exists a unique reduced root system Σ in V^* such that W^{aff} is the affine Weyl group of Σ, i. e. it is the subgroup of* $\operatorname{Aut} \mathcal{A}$ *generated by the reflections $s_{\alpha,k}$ for $(\alpha, k) \in \Sigma^{\mathrm{aff}} := \Sigma \times \mathbb{Z}$.*

Proof. See [BT72, Prop. 6.2.22]. Since φ_0 is discrete, the hyperplane arrangement $\mathfrak{H}$ is *locally finite*, i. e. for each compact subset $K \subseteq \mathcal{A}$ the set $\{H \in \mathfrak{H} \mid H \cap K \neq \emptyset\}$ is finite. As W^{aff} acts on $\mathfrak{H}$ and $W^{\mathrm{aff}} \cap V$ is a lattice of rank $\dim V$ in V, [Bou81, Ch. VI, §2.5, Prop. 8] shows that

$$\Sigma := \{\pm \alpha_H \in V^* \mid H \in \mathfrak{H}, \varphi_0 \in H\} \tag{1.5.1}$$

is the desired reduced root system, where α_H is defined as follows: Let $v_H \in V$ be a unit vector orthogonal to $H - \varphi_0$. Take $\lambda \in \mathbb{R}_{>0}$ minimal with $H' := H + \lambda v_H \in \mathfrak{H}$. Let $\alpha_H \in V^*$ be the linear form on V such that $H' = H_{\alpha_H,1}$. Then $\{\pm \alpha_H\}$ does not depend on the choice of v_H. □

Remark 1.33. We collect some observations regarding Proposition 1.32:

(a) For each $\alpha \in \Phi$ there exists a number $\varepsilon_\alpha \in \mathbb{N}$ with $\varepsilon_\alpha = \varepsilon_{-\alpha}$ and such that the map $\Phi \to \Sigma$, $\alpha \mapsto \varepsilon_\alpha \alpha$ is surjective and induces a bijection $\Phi_{\mathrm{red}} \cong \Sigma$ [Vig16, (39) ff].

(b) We have $\Gamma_\alpha = \varepsilon_\alpha^{-1} \mathbb{Z}$ for $\alpha \in \Phi_{\mathrm{red}}$. In particular, Γ_α is a group.

(c) Let Π be the basis of Σ corresponding to Δ under the bijection in (a). Then Proposition 1.31 reads

$$W^{\mathrm{aff}} \cap V = Q(\Sigma^\vee) := \left\{ \sum_{\alpha \in \Pi} k_\alpha \alpha^\vee \,\middle|\, k_\alpha \in \mathbb{Z} \text{ for } \alpha \in \Pi \right\} \quad \text{and}$$
$$\widetilde{W} \cap V \subseteq P(\Sigma^\vee) := \left\{ \sum_{\alpha \in \Pi} k_\alpha \varpi_\alpha \,\middle|\, k_\alpha \in \mathbb{Z} \text{ for } \alpha \in \Pi \right\}$$

(where $\{\varpi_\alpha \mid \alpha \in \Pi\}$ is the dual basis of Π in V). By Proposition 1.30 we have a semidirect product decomposition

$$W^{\mathrm{aff}} = Q(\Sigma^\vee) \rtimes W_0. \tag{1.5.2}$$

(d) The map $\Sigma^{\mathrm{aff}} \to \Phi^{\mathrm{aff}}$, $(\alpha, k) \mapsto a_{\alpha,k}$ is bijective. In view of Proposition 1.27 (iii) this map is $\widetilde{W}$-equivariant if we let $t_v w_0 \in \widetilde{W} \subseteq P(\Sigma^\vee) \rtimes W_0$ act on Σ^{aff} via

$$t_v w_0 \cdot (\alpha, k) = \big(w_0(\alpha), k - \langle w_0(\alpha), v\rangle\big). \tag{1.5.3}$$

(e) We will view $(\alpha, k) \in \Sigma^{\mathrm{aff}}$ also as a function $V \to \mathbb{R}$, $v \mapsto \langle(\alpha, k), v\rangle := \langle \alpha, v\rangle + k$. We define the *positive affine roots* by

$$\begin{aligned}\Sigma^{\mathrm{aff},+} &:= \Big\{(\alpha, k) \in \Sigma^{\mathrm{aff}} \,\Big|\, \langle \alpha, x - \varphi_0\rangle + k \geq 0\Big\} \\ &= \Sigma^+ \cup \Big\{(\alpha, k) \in \Sigma^{\mathrm{aff}} \,\Big|\, \alpha \in \Sigma, k > 0\Big\},\end{aligned}$$

where $x \in \mathfrak{C}$ is an arbitrary point. Accordingly, we define $\Sigma^{\mathrm{aff},-} := \Sigma^{\mathrm{aff}} \smallsetminus \Sigma^{\mathrm{aff},+}$.

Notation 1.34. To avoid confusion arising from the usage of the two root systems Φ_{red} and Σ we write $H_{(\alpha,k)}$ (resp. $U_{(\alpha,k)}$) instead of $H_{\alpha,k}$ (resp. $U_{\beta,\varepsilon_\beta^{-1}k}$), whenever $\beta \in \Phi_{\mathrm{red}}$, $\alpha = \varepsilon_\beta \beta$, and $k \in \mathbb{Z}$.

Lemma 1.35. *Let $n \in N$ with image w in $\widetilde{W}$, and let $(\alpha, k) \in \Sigma^{\mathrm{aff}}$. We have $wH_{(\alpha,k)} = H_{w\cdot(\alpha,k)}$ and*

$$nU_{(\alpha,k)}n^{-1} = U_{w\cdot(\alpha,k)}. \tag{1.5.4}$$

Proof. Write $w = t_v w_0$ with $v \in P(\Sigma^\vee)$ and $w_0 \in W_0$, and let $\beta \in \Phi_{\mathrm{red}}$ with $\alpha = \varepsilon_\beta \beta$. Using Proposition 1.27 (iii) we compute

$$\begin{aligned}nU_{(\alpha,k)}n^{-1} &= nU_{\beta,\varepsilon_\beta^{-1}k}n^{-1} = U_{w_0(\beta),\varepsilon_\beta^{-1}k - \langle w_0(\beta), v\rangle} \\ &= U_{w_0(\beta),\varepsilon_\beta^{-1}(k - \langle w_0(\varepsilon_\beta\beta), v\rangle)} = U_{w_0(\alpha), k - \langle w_0(\alpha), v\rangle} = U_{w\cdot(\alpha,k)}.\end{aligned}$$

As $H_{\alpha,k} = \partial a_{\alpha,k}$ and the map $\Sigma^{\mathrm{aff}} \to \Phi^{\mathrm{aff}}$ is $\widetilde{W}$-equivariant, it follows that $wH_{(\alpha,k)} = H_{w\cdot(\alpha,k)}$. □

Recall the fundamental alcove $\mathfrak{C}$ of $\mathcal{A}$ and its set of walls $\mathfrak{H}(\mathfrak{C})$. We denote by S^{aff} or $S(\mathfrak{C})$ the corresponding set of reflections in W^{aff}.

Proposition 1.36. *(i) The set S^{aff} generates W^{aff} and consists of elements of order two. Denote by $\ell\colon W^{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ the corresponding* length function, *i. e. given $w \in W^{\mathrm{aff}}$ we denote by $\ell(w)$ the least integer n such that there exist $s_1, \dots, s_n \in S^{\mathrm{aff}}$ with $w = s_1 \cdots s_n$. Then the following* deletion condition *is satisfied: given $w \in W^{\mathrm{aff}}$ and $s_1, \dots, s_n \in S^{\mathrm{aff}}$ with $w = s_1 \cdots s_n$ and $n > \ell(w)$, there exist $1 \leq i < j \leq n$ with $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$, where $\widehat{(\cdot)}$ denotes omission. In other words: $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ is a* Coxeter system.

(ii) *W^{aff} acts simply transitively on the set of alcoves of $\mathcal{A}$.*

(iii) *The topological closure $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ is a* fundamental domain *for the action of W^{aff} on $\mathcal{A}$, and for each $x \in \overline{\mathfrak{C}}$ the stabilizer $W^{\mathrm{aff}}_x := \{ w \in W^{\mathrm{aff}} \,|\, wx = x \}$ is generated by $S^{\mathrm{aff}}_x := \{ s \in S^{\mathrm{aff}} \,|\, sx = x \}$.*

(iv) *Let $w \in W^{\mathrm{aff}}$ and write $w = s_1 \cdots s_n$ with $s_1, \dots, s_n \in S^{\mathrm{aff}}$ and $n = \ell(w)$. We denote by $\mathfrak{H}_w$ the set of hyperplanes separating $\mathfrak{C}$ and $w\mathfrak{C}$. Let $\mathfrak{T}_w$ be the corresponding set of reflections. We have*

$$\mathfrak{H}_w = \{H_{s_1}, s_1 H_{s_2}, (s_1 s_2) H_{s_3}, \dots, (s_1 \cdots s_{n-1}) H_{s_n}\} \quad \textit{and}$$
$$\mathfrak{T}_w = \{s_1, s_1 s_2 s_1, (s_1 s_2) s_3 (s_2 s_1), \dots, (s_1 \cdots s_{n-1}) s_n (s_{n-1} \cdots s_1)\},$$

and moreover $\ell(w) = |\mathfrak{H}_w| = |\mathfrak{T}_w|$.

Proof. See [Gar97, 12.1, 12.2]. Given two alcoves $\mathfrak{D}$ and $\mathfrak{D}'$ of $\mathcal{A}$, a *gallery* from $\mathfrak{D}$ to $\mathfrak{D}'$ is a sequence $(\mathfrak{D}_0, \dots, \mathfrak{D}_n)$ of alcoves with $\mathfrak{D}_0 = \mathfrak{D}$, $\mathfrak{D}_n = \mathfrak{D}'$, and $\mathfrak{H}(\mathfrak{D}_i) \cap \mathfrak{H}(D_{i+1}) \neq \varnothing$ for $0 \leq i \leq n-1$; in this case either $\mathfrak{D}_i = \mathfrak{D}_{i+1}$ or $\mathfrak{H}(\mathfrak{D}_i) \cap \mathfrak{H}(\mathfrak{D}_{i+1})$ contains exactly one hyperplane, which is then said to be *crossed* by the gallery [Gar97, 12.1 Prop.]. The integer n is called the *length* of the gallery. We may thus define a distance function by letting $\ell(\mathfrak{D}, \mathfrak{D}')$ be the least integer $n \geq 0$ such that there exists a gallery of length n from $\mathfrak{D}$ to $\mathfrak{D}'$. Then $\ell(\mathfrak{D}, \mathfrak{D}')$ coincides with the number of hyperplanes separating $\mathfrak{D}$ and $\mathfrak{D}'$ [Gar97, 12.2 Lem.].

Let now W' be the subgroup of W^{aff} generated by S^{aff}. We use induction on the lengths of galleries to prove that W' acts transitively on the set of alcoves of $\mathcal{A}$. Let $\mathfrak{D}$ be some alcove and let H be a wall of $\mathfrak{D}$ separating $\mathfrak{C}$ and $\mathfrak{D}$. Let $\mathfrak{D}' := s_H(\mathfrak{D})$. Then we have $\ell(\mathfrak{C}, \mathfrak{D}') = \ell(\mathfrak{C}, \mathfrak{D}) - 1$, hence by induction hypothesis there exists some $w \in W'$ with $w\mathfrak{C} = \mathfrak{D}'$. Then $w^{-1}H$ is the unique wall of $\mathfrak{C}$ separating $\mathfrak{C}$ and $w^{-1}\mathfrak{D}$. Then $s := w^{-1} s_H w = s_{w^{-1}H} \in S^{\mathrm{aff}}$ and

$$ws\mathfrak{C} = w(w^{-1} s_H w)\mathfrak{C} = s_H w \mathfrak{C} = s_H \mathfrak{D}' = \mathfrak{D}.$$

Since $ws \in W'$, this shows that W' acts transitively on the set of alcoves.

Given any $s \in S(\mathfrak{H})$, let $\mathfrak{D}$ be some alcove with $H_s \in \mathfrak{H}(\mathfrak{D})$. By what we have just proved, there exists $w \in W'$ with $w\mathfrak{C} = \mathfrak{D}$. But then $w^{-1}H_s$ is a wall of $\mathfrak{C}$, i. e. $w^{-1}sw \in S^{\mathrm{aff}}$. This implies $s \in W'$. As W^{aff} is generated by $S(\mathfrak{H})$, we conclude $W' = W^{\mathrm{aff}}$, i. e. W^{aff} is generated by S^{aff}.

We now prove the deletion condition. Let $w = s_1 \cdots s_n$ be some decomposition with $s_1 \dots, s_n \in S^{\mathrm{aff}}$ and $n > \ell(w)$. Consider the sequence

$$\Gamma := (\mathfrak{C}, s_1\mathfrak{C}, s_1 s_2 \mathfrak{C}, \dots, s_1 \cdots s_n \mathfrak{C}).$$

Since H_{s_i} is the common wall of $\mathfrak{C}$ and $s_i\mathfrak{C}$, it follows that $s_1 \cdots s_{i-1} H_{s_i}$ is the common wall of $s_1 \cdots s_{i-1}\mathfrak{C}$ and $s_1 \cdots s_i \mathfrak{C}$. Consequently, Γ is a gallery from $\mathfrak{C}$ to $w\mathfrak{C}$ of length n. In particular, $\ell(w) \geq \ell(\mathfrak{C}, w\mathfrak{C})$. If $n > \ell(\mathfrak{C}, w\mathfrak{C})$, this means that Γ crosses some hyperplane twice. Hence, there exist $1 \leq i < j \leq n$ with

$$H_{(s_1 \cdots s_{i-1}) s_i (s_{i-1} \cdots s_1)} = s_1 \cdots s_{i-1} H_{s_i} = s_1 \cdots s_{j-1} H_{s_j} = H_{(s_1 \cdots s_{j-1}) s_j (s_{j-1} \cdots s_1)},$$

i. e. $(s_1 \cdots s_{i-1})s_i(s_{i-1} \cdots s_1) = (s_1 \cdots s_{j-1})s_j(s_{j-1} \cdots s_1)$. Multiplying from the right with the element $s_1 \cdots \widehat{s_j} \cdots s_n$ yields $s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n = s_1 \cdots s_n$. This proves the deletion condition and $\ell(w) = \ell(\mathfrak{C}, w\mathfrak{C})$. Moreover, this discussion shows that each minimal gallery from $\mathfrak{C}$ to $w\mathfrak{C}$ crosses the hyperplanes separating $\mathfrak{C}$ and $w\mathfrak{C}$ (and only these) exactly once. In particular, $\mathfrak{C} = w\mathfrak{C}$ implies $\ell(w) = 0$ and hence $w = 1$. Thus, the action of W^{aff} an the alcoves of $\mathcal{A}$ is free.

It remains to prove (iii). It follows from (ii) that $\mathcal{A}$ is covered by the W^{aff}-translates of $\overline{\mathfrak{C}}$. To prove that $\overline{\mathfrak{C}}$ is a fundamental domain, it therefore remains to prove that, given $x, y \in \overline{\mathfrak{C}}$ and $w \in W^{\mathrm{aff}}$ with $wx = y$, we actually have $x = y$. We prove this and the fact that W_x^{aff} is generated by S_x^{aff} by induction on $\ell(w)$. Write $w = s_1 \cdots s_n$ with $s_1, \dots, s_n \in S^{\mathrm{aff}}$ and $n = \ell(w)$. Then the gallery

$$\Gamma := (\mathfrak{C}, s_1\mathfrak{C}, s_1s_2\mathfrak{C}, \dots, s_1 \cdots s_n\mathfrak{C})$$

from $\mathfrak{C}$ to $w\mathfrak{C}$ has minimal length and crosses H_{s_1}. Hence H_{s_1} separates $\mathfrak{C}$ and $w\mathfrak{C}$. By definition we must have $\overline{\mathfrak{C}} \cap w\overline{\mathfrak{C}} \subseteq H_{s_1}$. Since $wx = y \in \overline{\mathfrak{C}} \cap w\overline{\mathfrak{C}}$, we deduce that s_1 fixes y. Now, we have $\ell(s_1w) < \ell(w)$ and $(s_1w)x = s_1y = y$, so the induction hypothesis implies $x = y$.

Letting $x = y$, i. e. $w \in W_x^{\mathrm{aff}}$, the above argument shows $s_1 \in S_x^{\mathrm{aff}}$ and, by induction hypothesis, $s_1w \in \langle S_x^{\mathrm{aff}} \rangle \subseteq W_x^{\mathrm{aff}}$. Hence, W_x^{aff} is generated by S_x^{aff}. □

As a Coxeter group, W^{aff} comes with a certain partial order, called the *Bruhat order*, which is characterized in the following proposition.

Proposition 1.37. *Given $w, w' \in W^{\mathrm{aff}}$, the following statements are equivalent:*

(i) There exist $t_1, \dots, t_n \in S(\mathfrak{H})$ with $\ell(w't_1 \cdots t_i) > \ell(w't_1 \cdots t_{i-1})$ for all $1 \le i \le n$ and $w = w't_1 \cdots t_n$.

(ii) There exist $t_1, \dots, t_n \in S(\mathfrak{H})$ with $\ell(t_i \cdots t_1w') > \ell(t_{i-1} \cdots t_1w')$ for all $1 \le i \le n$ and $w = t_n \cdots t_1w'$.

(iii) Write $w = s_1 \cdots s_n$ with $s_1, \dots, s_n \in S^{\mathrm{aff}}$ and $n = \ell(w)$. Then there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ with $w' = s_{i_1} \cdots s_{i_k}$.

We write $w' \le w$ if any of the above statements holds.

Proof. The equivalence of (i) and (ii) follows from the fact that $S(\mathfrak{H})$ is stable under conjugation by elements of W^{aff}. For the equivalence of (i) and (iii) see [Gar97, 1.8, Thm.]. □

Remark 1.38. It is well-known (see e. g. [Gar97, 1.8, Cor. 2]) that in (i) above we may choose $t_1, \dots, t_n \in S(\mathfrak{H})$ such that $\ell(w't_1 \cdots t_i) = \ell(w't_1 \cdots t_{i-1}) + 1$ for all $1 \le i \le n$. An analogous statement holds for (ii).

1.6. The adjoint building, parahoric subgroups, and Weyl groups

We keep the notations of section 1.4.

Given $x \in \mathcal{A}$ and $\alpha \in \Phi$, we let

$$r_x(\alpha) := \inf\{r \in \Gamma_\alpha \mid \langle \alpha, x - \varphi_0 \rangle + r \geq 0\} = \inf\left\{r \in \Gamma_\alpha \,\middle|\, x \in a_{\alpha,r}\right\}.$$

This definition only depends on the equivalence class of x. If $\mathfrak{F}$ is the face containing x, we also write $r_{\mathfrak{F}}(\alpha) := r_x(\alpha)$. If the face $\mathfrak{F}$ is contained in $H_{\alpha,r}$ for some $\alpha \in \Phi$ and $r \in \Gamma_\alpha$, then we have $r_{\mathfrak{F}}(\alpha) = r = -r_{\mathfrak{F}}(-\alpha)$. If $\alpha \in \Phi^+_{\mathrm{red}}$, then $r_{\mathfrak{C}}(\alpha) = 0$ and $r_{\mathfrak{C}}(-\alpha) = \varepsilon_\alpha^{-1}$ (cf. Remark 1.33 (b)).

We let U_x be the subgroup of G generated by the $U_{\alpha,r_x(\alpha)}$ for $\alpha \in \Phi$. Denote by N_x the stabilizer of x in N. These groups only depend on the face $\mathfrak{F}$ containing x, so we may write $U_{\mathfrak{F}}$ (resp. $N_{\mathfrak{F}}$) instead of U_x (resp. N_x).

Lemma 1.39. *Let $x \in \mathcal{A}$. We have $nU_xn^{-1} = U_{\nu(n)(x)}$ and $nN_xn^{-1} = N_{\nu(n)(x)}$ for all $n \in N$. In particular, U_x is normalized by N_x and hence $P_x := N_xU_x$ is a group with $nP_xn^{-1} = P_{\nu(n)(x)}$ for $n \in N$.*

Proof. Let $n \in N_x$ and $\alpha \in \Phi$. If $r \in \Gamma_\alpha$ is such that $x \in a_{\alpha,r}$, then Proposition 1.27 (iii) shows

$$x = \nu(n).x \in \nu(n)(a_{\alpha,r}) = a_{w(\alpha),r-\langle w(\alpha),n.\varphi_0-\varphi_0\rangle},$$

where $w := {}^{\mathrm{v}}\nu(n) \in W_0$. Therefore, $r_x(\alpha) - \langle w(\alpha), n.\varphi_0 - \varphi_0\rangle \geq r_x\big(w(\alpha)\big)$, and again from Proposition 1.27 (iii) it follows that $nU_{\alpha,r_x(\alpha)}n^{-1} \subseteq U_{w(\alpha),r_x(w(\alpha))}$. By symmetry we actually have $nU_{\alpha,r_x(\alpha)}n^{-1} = U_{w(\alpha),r_x(w(\alpha))}$, from which we conclude $nU_xn^{-1} = U_{\nu(n)(x)}$ for all $n \in N$. The other assertions are trivial. □

Definition 1.40. We define a relation $\sim$ on $G \times \mathcal{A}$ by letting $(g, x) \sim (h, y)$ if there exists $n \in N$ with $\nu(n)(x) = y$ and $g^{-1}hn \in P_x$. Using Lemma 1.39 it is easily checked that $\sim$ is an equivalence relation.

We call $\mathcal{B} := G \times \mathcal{A}/\sim$ the *adjoint building* of G.

Proposition 1.41. *Consider the building $\mathcal{B}$ of G.*

- *(i) G acts on $\mathcal{B}$ via $g \cdot (g', x) := (gg', x)$ for $x \in \mathcal{A}$, $g, g' \in G$.*
- *(ii) The map $\mathcal{A} \to \mathcal{B}$, $x \mapsto (1, x)$ is an N-equivariant embedding; we may thus identify the apartment $\mathcal{A}$ with its image in $\mathcal{B}$.*
- *(iii) For each $x \in \mathcal{A}$ the group P_x is the stabilizer of x in G.*
- *(iv) For each $\alpha \in \Phi$ and $r \in \Gamma_\alpha$ the group $U_{\alpha,r}$ fixes the affine root $a_{\alpha,r}$.*

Proof.
- (i) We verify that the above action is well-defined. Let $g, g', h' \in G$, $x, y \in \mathcal{A}$ with $(g', x) \sim (h', y)$. We show $(gg', x) \sim (gh', y)$. As $(g', x) \sim (h', y)$ there exists $n \in N$ with $y = \nu(n)(x)$ and $g'^{-1}h'n \in P_x$. But then also $(gg')^{-1}(gh')n = g'^{-1}h'n \in P_x$, i.e. $(gg', x) \sim (gh', y)$. Therefore, the action of G on $G \times \mathcal{A}$ descends to an action on $\mathcal{B}$.
- (ii) If we have $(1, x) \sim (1, y)$, there exists $n \in N$ with $\nu(n)(x) = y$ and $n \in P_x$. From $P_x \cap N = N_x$ [BT72, (7.1.8)] we conclude $y = \nu(n)(x) = x$, i.e. the map in question is injective. Given $x \in \mathcal{A}$ and $n \in N$, we have $n \cdot (1, x) = (n, x) \sim \big(1, \nu(n)(x)\big)$, hence the embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ is N-equivariant.

(iii) Given $x \in \mathcal{A}$ and $g \in G$ with $g \cdot (1, x) = (g, x) \sim (1, x)$, there exists $n \in N$ with $\nu(n)(x) = x$ and $g^{-1}n \in P_x$. But we have $n \in N_x \subseteq P_x$, and thus $g \in P_x$. Conversely, it is clear that P_x stabilizes x.

(iv) Let $x \in a_{\alpha,r}$. By definition we have $U_{\alpha,r} \subseteq U_x \subseteq P_x$ and hence $u \cdot (1, x) = (u, x) \sim (1, x)$ for all $u \in U_{\alpha,r}$. □

In view of Proposition 1.41 it makes sense to define the apartments of $\mathcal{B}$ as the subsets of the form $g \cdot \mathcal{A}$ together with $g \cdot \mathcal{F}$ as the set of faces, $g \in G$. The faces of $\mathcal{A}$ inside $\mathcal{A} \cap g \cdot \mathcal{A}$ are then also the faces of $g \cdot \mathcal{A}$, $g \in G$. Therefore, this defines unambiguously the notion of *faces of* $\mathcal{B}$.

We now come to the definition of *parahoric subgroups* of G. Let F^{sep} be a separable closure of F and F^{unr} the maximal unramified extension of F in F^{sep}. Denote by $\sigma \in \mathrm{Gal}(F^{\mathrm{unr}}/F)$ the Frobenius automorphism. Let $\widehat{\mathbf{G}}$ be the Langlands dual of $\mathbf{G}$ with center $\mathbf{Z}(\widehat{\mathbf{G}})$. Then Kottwitz defines in [Kot97, 7.1 to 7.4] a functorial surjection

$$\kappa_G \colon G \longrightarrow X^*\big(\mathbf{Z}(\widehat{\mathbf{G}})\big)^{\sigma}_{\mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{unr}})} \tag{1.6.1}$$

from G to the σ-invariants of the $\mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{unr}})$-coinvariants of the group $X^*\big(\mathbf{Z}(\widehat{\mathbf{G}})\big)$ of F^{sep}-characters of $\mathbf{Z}(\widehat{\mathbf{G}})$.

Definition 1.42. Given a face $\mathfrak{F}$ of $\mathcal{B}$, we call the subgroup $K_{\mathfrak{F}} := \operatorname{Ker} \kappa_G \cap P_{\mathfrak{F}}$ a *parahoric subgroup* of G. Its pro-p radical $K_{\mathfrak{F}}(1)$ is called a *pro-p parahoric subgroup* of G. If $\mathfrak{F}$ is an alcove, then $K_{\mathfrak{F}}$ (resp. $K_{\mathfrak{F}}(1)$) is called an *Iwahori subgroup* (resp. a *pro-p Iwahori subgroup*) of G.

We fix the maximal parahoric subgroup $K := K_{\{\varphi_0\}}$ and the Iwahori subgroup $I := K_{\mathfrak{C}} \subseteq K$. We remark here that K is a compact open subgroup of G (although in general not maximal with this property) and satisfies $K \cap U_\alpha = U_{\alpha,0}$ for all $\alpha \in \Phi$ [Vig16, (51)].

Recall the centralizer $\mathbf{Z}$ of the maximal split torus $\mathbf{T}$ in $\mathbf{G}$.

Lemma 1.43. *The subgroup* $Z_0 := \operatorname{Ker} \kappa_Z$ *of* Z *is compact open. We have* $Z_0 = Z \cap K = Z \cap I$ *and* $Z_0(1) := Z \cap K(1) = Z \cap I(1)$ *is the pro-p-radical of* Z_0. *The group* N *normalizes* Z_0 *and* $Z_0(1)$. *Moreover, the multiplication map induces a homeomorphism*

$$\prod_{\alpha \in \Sigma^-} U_{(\alpha,1)} \times Z_0(1) \times \prod_{\alpha \in \Sigma^+} U_{(\alpha,0)} \xrightarrow{\sim} I(1). \tag{1.6.2}$$

Proof. From [Vig16, Prop. 3.15] (or rather [HR09, Lem. 4.2.1]) it follows that $Z_0 = Z \cap K = Z \cap I$ and hence Z_0 is compact open. We have $Z_0 = Z \cap \operatorname{Ker} \kappa_G \cap P_{\varphi_0}$. Clearly, N normalizes $\operatorname{Ker} \kappa_G$ and Z. Each element in $Z \cap P_{\varphi_0}$ acts trivially on $\mathcal{A}$, as it acts by a translation fixing φ_0 (see Proposition 1.41, (iii)). But then we have $n(Z \cap P_{\varphi_0})n^{-1} \subseteq Z \cap P_{\varphi_0}$. It follows that N normalizes Z_0. The same is true for $Z_0(1)$, as it is a characteristic subgroup of Z_0. For (1.6.2) we refer to [Vig16, Cor. 3.20]. □

Definition 1.44. We call $W := N/Z_0$ the *Iwahori-Weyl group* and $W(1) := N/Z_0(1)$ the *pro-p Iwahori-Weyl group*.

Denoting $\Lambda := Z/Z_0$ and $\Lambda(1) := Z/Z_0(1)$ we have the exact sequences

$$1 \longrightarrow Z_\kappa \longrightarrow W(1) \longrightarrow W \longrightarrow 1,$$

where $Z_\kappa := Z_0/Z_0(1)$, and

$$0 \longrightarrow \Lambda \longrightarrow W \longrightarrow W_0 \longrightarrow 1. \tag{1.6.3}$$

Given a subset $X \subseteq W$, we denote by $X(1)$ its preimage in $W(1)$ under the canonical projection $W(1) \to W$.

The group Λ is a finitely generated abelian group with finite torsion and the same rank as $X_*(T)$ [HR09, Thm. 1.0.1]. It is thus denoted additively. To prevent confusion, we adopt the exponential notation when viewing Λ as a subgroup of W, i. e. we write e^λ for $\lambda \in \Lambda$. Moreover, the sequence (1.6.3) splits; thus, we obtain a semidirect product decomposition

$$W = \Lambda \rtimes W_0. \tag{1.6.4}$$

In particular, W_0 acts on Λ via $w(\lambda) := we^\lambda w^{-1}$ for $w \in W_0$, $\lambda \in \Lambda$. The group $\Lambda(1)$ is not abelian in general.

Remark 1.45. By [Vig16, Rmk. 3.37, Def. 3.32] the image of $\operatorname{Ker} \kappa_G \cap N$ under $N \twoheadrightarrow W$ identifies with W^{aff}. In particular, K contains representatives for the finite Weyl group W_0. In other words, the inclusion $N \hookrightarrow G$ induces a surjective map $Z \twoheadrightarrow K\backslash G/K$.

Notice that the action of N on $\mathscr{A}$ (Proposition 1.27) induces an action of W and $W(1)$ on $\mathscr{A}$. By Remark 1.33, (c) we have

$$Q(\Sigma^\vee) \subseteq \nu(\Lambda) = \nu\big(\Lambda(1)\big) \subseteq P(\Sigma^\vee) = \{v \in V \mid \langle \alpha, v\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Pi\}, \tag{1.6.5}$$

where $\Pi \subseteq \Sigma$ is the basis corresponding to $\Delta \subseteq \Phi$. Given $\lambda \in \Lambda$ and $w \in W_0$, it holds

$$w\big(\nu(\lambda)\big) = \nu\big(w(\lambda)\big), \tag{1.6.6}$$

because we have $w(\varphi_0) = \varphi_0$ and $w(\lambda) = we^\lambda w^{-1} \in \Lambda$, and hence

$$\begin{aligned} w\big(\nu(\lambda)\big) &= w\big(e^\lambda(\varphi_0) - \varphi_0\big) = we^\lambda(\varphi_0) - w(\varphi_0) \\ &= (we^\lambda w^{-1})\big(w(\varphi_0)\big) - w(\varphi_0) = (we^\lambda w^{-1})(\varphi_0) - \varphi_0 = \nu\big(w(\lambda)\big). \end{aligned}$$

Proposition 1.46 (Bruhat decompositions)**.** *The inclusion $N \hookrightarrow G$ induces the bijections*

$$W \cong I\backslash G/I \quad \textit{and} \quad W(1) \cong I(1)\backslash G/I(1).$$

Proof. See [Vig16, Prop. 3.35]. □

Let $\Omega := \{nZ_0 \in W \mid \nu(n)(\mathfrak{C}) = \mathfrak{C}\}$. Then by [Vig16, (57)] we have

$$W = W^{\mathrm{aff}} \rtimes \Omega$$

and $W(1) = W^{\mathrm{aff}}(1)\Omega(1)$ with $W^{\mathrm{aff}}(1) \cap \Omega(1) = Z_\kappa$.

Lemma 1.47. *(i) The length function ℓ on W^{aff} extends to a length function, also denoted ℓ, on W such that $\Omega = \{w \in W \mid \ell(w) = 0\}$. Under the canonical projection $W(1) \to W$ the length function ℓ inflates to a length function, again denoted by ℓ, on $W(1)$ with $\Omega(1) = \{w \in W(1) \mid \ell(w) = 0\}$.*

(ii) The Bruhat order $\leq$ on W^{aff} extends to a partial order, also denoted by $\leq$, on W by letting $wu \leq w'u'$ if and only if $w \leq w'$ and $u = u'$, where $w, w' \in W^{\mathrm{aff}}$ and $u, u' \in \Omega$. Under the canonical projection $W(1) \to W$ the partial order $\leq$ on W inflates to a preorder $\leq$ on $W(1)$. In any case $\leq$ is called the Bruhat order of W, resp. $W(1)$.

Proof. (i) follows from the fact that Ω acts by conjugation on S^{aff}: if $u \in \Omega$ and $s \in S^{\mathrm{aff}}$, then H_s is a wall of $\mathfrak{C}$ and usu^{-1} is the reflection in the wall $H_{usu^{-1}} = uH_s$ of $\mathfrak{C}$. Since also $usu^{-1} \in W^{\mathrm{aff}}$, it follows that $usu^{-1} \in S^{\mathrm{aff}}$.

(ii) is straight-forward. Notice that the Bruhat order on $W(1)$ is not antisymmetric whenever $Z_0 \neq Z_0(1)$. □

We end this section by describing the length of $W(1)$ and W more explicitly as in [Vig16, Sec. 5.1]. Recall the root system Σ attached to the affine Weyl group W^{aff} (1.5.1). We choose a point $x \in \mathfrak{C}$ and let $\Sigma^+ = \{\alpha \in \Sigma \mid \langle \alpha, x - \varphi_0\rangle > 0\}$ be a choice of positive roots. Notice that $\Pi = \left\{\alpha \in \Sigma^+ \,\middle|\, H_{s_{\alpha,\alpha^\vee}} \text{ is a wall of } \mathfrak{C}\right\}$ is the unique basis contained in Σ^+; it is the one corresponding to $\Delta \subseteq \Phi$. We identify Σ with the subset $\Sigma \times \{0\}$ of $\Sigma^{\mathrm{aff}} = \Sigma \times \mathbb{Z}$. Recall also the action of W and $W(1)$ on Σ^{aff} (1.5.3).

Definition 1.48. Let $w \in W$ and let $\tilde{w} \in W(1)$ be a lift of w. For each $\beta \in \Sigma$ we define $\ell_\beta(\tilde{w}) = \ell_\beta(w) \in \mathbb{Z}$ by requiring

$$\ell_\beta(w) < \langle \beta, w(x) - \varphi_0\rangle < \ell_\beta(w) + 1.$$

By definition of the equivalence relation on $\mathscr{A}$, the integer $\ell_\beta(w)$ does not depend on $x \in \mathfrak{C}$. Notice that $\langle \beta, w(x) - \varphi_0\rangle = \langle w^{-1}(\beta), x - \varphi_0\rangle$, where $w^{-1}\beta \in \Sigma^{\mathrm{aff}}$: if we write $w = t_v w_0$ with $v \in P(\Sigma^\vee)$ and $w_0 \in W_0$, we have $w(x) - \varphi_0 = w_0(x - \varphi_0) + v$ and hence

$$\begin{aligned}
\langle \beta, w(x) - \varphi_0\rangle &= \langle \beta, w_0(x - \varphi_0)\rangle + \langle \beta, v\rangle \\
&= \langle w_0^{-1}(\beta), x - \varphi_0\rangle + \langle \beta, v\rangle \\
&= \langle (w_0^{-1}(\beta), \langle \beta, v\rangle), x - \varphi_0\rangle \\
&= \langle w_0^{-1} t_{-v} \cdot (\beta, 0), x - \varphi_0\rangle \\
&= \langle w^{-1}(\beta), x - \varphi_0\rangle.
\end{aligned}$$

Lemma 1.49. *Let $w \in W$, $\beta \in \Sigma^+$, and $k \in \mathbb{Z}$. Then $H_{(\beta,k)}$ separates $\mathfrak{C}$ and $w\mathfrak{C}$ if and only if either*

- ⋄ *$\ell_\beta(w) \leq -1$ and $0 \leq k \leq -\ell_\beta(w) - 1$, or*
- ⋄ *$\ell_\beta(w) \geq 1$ and $-\ell_\beta(w) \leq k \leq -1$.*

Proof. See [Vig16, Lem. 5.6]. Assume that $H_{(\beta,k)}$ separates $\mathfrak{C}$ and $w\mathfrak{C}$. Recall the point $x \in \mathfrak{C}$. There are two cases to consider:

Case 1: $\langle\beta, x-\varphi_0\rangle + k > 0$ and $\langle\beta, w(x)-\varphi_0\rangle + k < 0$. The second inequality and the definition of $\ell_\beta(w)$ show $\ell_\beta(w)+1 \leq -k$. Since $\beta \in \Sigma^+$ we have $0 < \langle\beta, x-\varphi_0\rangle < 1$. Together with the first inequality this shows $0 < \langle\beta, x-\varphi_0\rangle + k < 1+k$ and hence $0 \leq k$. Taken together, we obtain $0 \leq k \leq -\ell_\beta(w)-1$, and hence also $\ell_\beta(w) \leq -1$.

Case 2: $\langle\beta, x-\varphi_0\rangle + k < 0$ and $\langle\beta, w(x)-\varphi_0\rangle + k > 0$. Similar to the first case, we have $\ell_\beta(w) \geq -k$ and $k+1 \leq 0$. Therefore, $-\ell_\beta(w) \leq k \leq -1$ and hence also $\ell_\beta(w) \geq 1$.

The converse holds, because all the arguments are reversible. □

Proposition 1.50. *Let $w \in W$ or $w \in W(1)$. Then we have $\ell(w) = \sum_{\beta\in\Sigma^+}|\ell_\beta(w)|$.*

Proof. See [Vig16, Prop. 5.7]. Let $w \in W^{\mathrm{aff}}$. By Proposition 1.36, (iv) we have $\ell(w) = |\mathfrak{H}_w|$, where $\mathfrak{H}_w$ denotes the set of hyperplanes separating $\mathfrak{C}$ and $w\mathfrak{C}$. As the map $\Sigma^+\times\mathbb{Z} \to \mathfrak{H}$, $(\beta,k) \mapsto H_{(\beta,k)}$ is injective (even bijective), the assertion follows from Lemma 1.49. Because of $\Omega\mathfrak{C} = \mathfrak{C}$ we have $\ell(wu) = \ell(w)$ and $\ell_\beta(wu) = \ell_\beta(w)$ for all $u \in \Omega$. Hence the proposition is true for $w \in W$. By definition of ℓ and ℓ_β it then also follows for $w \in W(1)$. □

Lemma 1.51. *Let $\beta \in \Sigma$, $\lambda \in \Lambda$ and $w \in W_0$. Then we have*

$$\ell_\beta(e^\lambda w) = \begin{cases} \langle\beta, \nu(\lambda)\rangle, & \textit{if } \beta \in w(\Sigma^+), \\ \langle\beta, \nu(\lambda)\rangle - 1, & \textit{if } \beta \in w(\Sigma^-). \end{cases}$$

Proof. See [Vig16, Lem. 5.9 2)]. Notice that $(e^\lambda w)(x) - \varphi_0 = w(x-\varphi_0) + \nu(\lambda)$, because of $w(\varphi_0) = \varphi_0$. Since also $\langle\beta, w(x-\varphi_0)\rangle = \langle w^{-1}(\beta), x-\varphi_0\rangle$, we have

$$\ell_\beta(e^\lambda w) < \langle w^{-1}(\beta), x-\varphi_0\rangle + \langle\beta,\nu(\lambda)\rangle < \ell_\beta(e^\lambda w) + 1.$$

If $w^{-1}(\beta) \in \Sigma^+$, then from $0 < \langle w^{-1}(\beta), x-\varphi_0\rangle < 1$ we infer $\ell_\beta(e^\lambda w) = \langle\beta,\nu(\lambda)\rangle$. If, however, $w^{-1}(\beta) \in \Sigma^-$, then from $-1 < \langle w^{-1}(\beta), x-\varphi_0\rangle < 0$ we infer $\ell_\beta(e^\lambda w) = \langle\beta,\nu(\lambda)\rangle - 1$. □

Lemma 1.51 and Proposition 1.50 yield the following length formula:

$$\ell(e^\lambda w) = \sum_{\alpha\in\Sigma^+\cap w(\Sigma^+)} \big|\langle\alpha,\nu(\lambda)\rangle\big| + \sum_{\alpha\in\Sigma^+\cap w(\Sigma^-)} \big|\langle\alpha,\nu(\lambda)\rangle - 1\big|, \tag{1.6.7}$$

for all $w \in W_0$, $\lambda \in \Lambda$.

1.7. Orientations

We keep the notation of the previous sections. We recall the apartment $\mathscr{A}$, its set of hyperplanes $\mathfrak{H}$ and the Iwahori-Weyl group W (resp. the pro-p Iwahori-Weyl group $W(1)$), which acts on $\mathscr{A}$ and on $\mathfrak{H}$.

Definition 1.52. [Gör07, Def. 2.3.1]. An *orientation* o of $(\mathscr{A}, \mathfrak{H})$ is a choice of a *positive half-space* for each hyperplane $H \in \mathfrak{H}$, denoted by $H_{o,+}$, such that either

(1) for each finite subset $\mathfrak{H}' \subseteq \mathfrak{H}$ the intersection $\bigcap_{H \in \mathfrak{H}'} H_{o,+}$ is non-empty, or
(2) for each finite subset $\mathfrak{H}' \subseteq \mathfrak{H}$ the intersection $\bigcap_{H \in \mathfrak{H}'} H_{o,-}$ is non-empty, where we write $H_{o,-} := \overline{\mathscr{A} \smallsetminus H_{o,+}}$ for $H \in \mathfrak{H}$.

Remark 1.53. The group W (and hence, by inflation, also $W(1)$) acts from the right on the set of orientations of $(\mathscr{A}, \mathfrak{H})$ as follows: let o be an orientation and $w \in W$. Then

$$H_{o \bullet w,+} := w^{-1}\big((wH)_{o,+}\big), \qquad \text{for } H \in \mathfrak{H}$$

defines a new orientation $o \bullet w$ of $(\mathscr{A}, \mathfrak{H})$, and we have $o \bullet 1 = o$ and

$$\begin{aligned} H_{o \bullet (vw),+} &= (vw)^{-1}\big(((vw)H)_{o,+}\big) = w^{-1}\big(v^{-1}\big((v(wH))_{o,+}\big)\big) \\ &= w^{-1}\big((wH)_{o \bullet v,+}\big) = H_{(o \bullet v) \bullet w,+}, \qquad \text{for } v, w \in W. \end{aligned}$$

Definition 1.54. [Vig16, Def. 5.18]. Let o be an orientation of $(\mathscr{A}, \mathfrak{H})$. We define the map

$$\varepsilon_o \colon W \times S^{\mathrm{aff}} \longrightarrow \{\pm 1\}, \quad (w, s) \longmapsto \begin{cases} 1, & \text{if } w\mathfrak{C} \subseteq (H_{wsw^{-1}})_{o,-} \\ -1, & \text{if } w\mathfrak{C} \subseteq (H_{wsw^{-1}})_{o,+}. \end{cases} \tag{1.7.1}$$

Notice that $w\mathfrak{C} \subseteq (H_{wsw^{-1}})_{o,+}$ if and only if $\mathfrak{C} \subseteq (H_s)_{o \bullet w,+}$. The map ε_o inflates to a map $W(1) \times S^{\mathrm{aff}}(1) \to \{\pm 1\}$, which is again denoted by ε_o.

We say a gallery $(\mathfrak{C}_0, \dots, \mathfrak{C}_n)$ crosses $H \in \mathfrak{H}$ in *o-positive direction* if there exists $0 \le r \le n-1$ such that $\mathfrak{C}_r \subseteq H_{o,-}$ and $\mathfrak{C}_{r+1} \subseteq H_{o,+}$. Given $(w, s) \in W \times S^{\mathrm{aff}}$, the gallery $(w\mathfrak{C}, ws\mathfrak{C})$ crosses $H_{wsw^{-1}}$ in o-positive direction if $\varepsilon_o(w, s) = 1$ and in o-negative direction if $\varepsilon_o(w, s) = -1$. We say that $H_{wsw^{-1}}$ is crossed in $\varepsilon_o(w, s)$-direction.

Example 1.55. Recall the fundamental alcove $\mathfrak{C}$ of $\mathscr{A}$. Choosing for each hyperplane of $\mathfrak{H}$ the half-space containing $\mathfrak{C}$ as the *negative* half-space, we obtain an orientation of $(\mathscr{A}, \mathfrak{H})$, called the *trivial orientation*. We also say that it is oriented away from the fundamental alcove $\mathfrak{C}$: each minimal gallery from $\mathfrak{C}$ to $w\mathfrak{C}$, for $w \in W$, crosses the hyperplanes in $\mathfrak{H}_w$ in o-positive direction.

1.8. Parabolic subgroups, Levi subgroups, and positive elements

We describe in this section the (standard) parabolic subgroups and their (standard) Levi subgroups of a connected reductive group $\mathbf{G}$. The main reference is [OV18, Sec. 2.4].

We recall the setup developed in the previous sections: let $\mathbf{G}$ be a connected reductive group over F, $\mathbf{T}$ a maximal F-split torus, and $\Phi := \Phi(\mathbf{G}, \mathbf{T})$ the associated root system. Fix a basis Δ of Φ. Let $\mathbf{N} := \mathbf{N}_{\mathbf{G}}(\mathbf{T})$ be the normalizer and $\mathbf{Z} := \mathbf{Z}_{\mathbf{G}}(\mathbf{T})$ be the centralizer of $\mathbf{T}$. Let $\mathbf{U}_\alpha$ be the root group associated with $\alpha \in \Phi$. We fixed a generating root group datum $\big(Z, (U_\alpha)_{\alpha \in \Phi}\big)$ of type Φ, together with a discrete special valuation φ_0 that is compatible with ω (see Theorem 1.22). Recall the $\mathbb{R}$-vector space $V := \big(X_*(T)/X_*(C)\big) \otimes_{\mathbb{Z}} \mathbb{R}$

(1.1.2), where $\mathbf{C}$ is the connected center of $\mathbf{G}$. We constructed an affine space $\mathscr{A}$ under V (1.4.1) together with its set of hyperplanes $\mathfrak{H}$ (1.4.3). Let $\mathfrak{C}$ be the unique alcove in $\mathscr{A}$, called the fundamental alcove, with $\varphi_0 \in \overline{\mathfrak{C}}$ and such that the set of positive roots Φ^+ coincides with $\{\alpha \in \Phi \mid \langle \alpha, x - \varphi_0 \rangle \geq 0\}$ for some fixed $x \in \mathfrak{C}$. Let $S^{\mathrm{aff}} = S(\mathfrak{C})$ be the set of reflections in the walls of $\mathfrak{C}$. Then S^{aff} generates the affine Weyl group W^{aff} of G. The subset $S = \{s_{\alpha,\alpha^\vee} \mid \alpha \in \Delta\} \subseteq S^{\mathrm{aff}}$ of reflections fixing φ_0 generates the Weyl group $W_0 = N/Z$. To W^{aff} we attached a unique reduced root system Σ (see Proposition 1.32). There is a surjective map $\Phi \to \Sigma$, $\alpha \mapsto \varepsilon_\alpha \alpha$, inducing a bijection $\Phi_{\mathrm{red}} \cong \Sigma$. Let Π be the basis of Σ corresponding to Δ. The parahoric subgroup $K := K_{\{\varphi_0\}}$ is a maximal compact open subgroup of G, and $I := K_{\mathfrak{C}}$ is an Iwahori subgroup with pro-p radical $I(1)$. We defined the Iwahori-Weyl group $W := N/Z_0$ and the pro-p Iwahori-Weyl group $W(1) := N/Z_0(1)$, where $Z_0 = I \cap Z$ and $Z_0(1) = I(1) \cap Z$. Let $\Lambda := Z/Z_0$ and $\Lambda(1) := Z/Z_0(1)$. Then we have $W = \Lambda \rtimes W_0 = W^{\mathrm{aff}} \rtimes \Omega$.

We choose a subset $J \subseteq \Delta$ and let $\Phi_J \subseteq \Phi$ be the root system generated by J with corresponding set of positive roots $\Phi_J^+ = \Phi_J \cap \Phi^+$. Write $\Psi := \Phi^+ \smallsetminus \Phi_J$ and denote by Ψ_{red} the reduced roots in Ψ. Consider the F-split subtorus $\mathbf{T}_J := \left(\bigcap_{\alpha \in \Phi_J} \operatorname{Ker} \alpha\right)^\circ$ of $\mathbf{T}$. Then $\mathbf{M}_J := \mathbf{Z}_{\mathbf{G}}(\mathbf{T}_J)$ is the (standard) Levi subgroup of the (standard) parabolic subgroup $\mathbf{P}_J$ of $\mathbf{G}$ [Bor91, 20.4 Prop.] with unipotent radical $\mathbf{U}_J := \prod_{\alpha \in \Psi_{\mathrm{red}}} \mathbf{U}_\alpha$. In particular, $\mathbf{M}_J$ is a connected reductive group over F with maximal F-split torus $\mathbf{T}$, with connected center $\mathbf{T}_J$, and with $\Phi_J = \Phi(\mathbf{M}_J, \mathbf{T})$. The opposite parabolic subgroup of $\mathbf{P}_J$ is $\mathbf{P}_J^{\mathrm{op}} := \mathbf{M}_J \mathbf{U}_J^{\mathrm{op}}$, where $\mathbf{U}_J^{\mathrm{op}} := \prod_{\alpha \in -\Psi_{\mathrm{red}}} \mathbf{U}_\alpha$. We have the decompositions $P_J = M_J \ltimes U_J$ and $P_J^{\mathrm{op}} = M_J \ltimes U_J^{\mathrm{op}}$.

From Example 1.5 (b) it follows that $(Z, (U_\alpha)_{\alpha \in \Phi_J})$ is a generating root group datum of type Φ_J of M_J, and that the restriction of φ_0 to $(Z, (U_\alpha)_{\alpha \in \Phi_J})$ is a discrete, special valuation $\varphi_{0,J}$ that is compatible with ω. We may define the same objects for M_J as we did above for G, and name them by attaching the index J. We have $N_J = N \cap M_J$, hence the Weyl group $W_{0,J}$ of M_J is contained in W_0. It is clear that the Iwahori-Weyl group W_J of M_J is contained in W and coincides with $\Lambda \rtimes W_{0,J}$. The pro-p Iwahori-Weyl group $W_J(1)$ of M_J coincides with the preimage of W_J under the projection map $W(1) \twoheadrightarrow W$; in this regard the notation is consistent. Notice that the $\mathbb{R}$-vector space $V_J := (X_*(T)/X_*(T_J)) \otimes_{\mathbb{Z}} \mathbb{R}$ is a quotient of V and the W_0-invariant pairing on V descends to a $W_{0,J}$-invariant pairing on V_J. By construction we have $\Phi_J = V_J^* \cap \Phi$.

Lemma 1.56. *Let $\mathscr{A}_J = \varphi_{0,J} + V_J$ be the affine apartment, Φ_J^{aff} the set of affine roots, and $\mathfrak{H}_J$ the system of hyperplanes associated with M_J.*

(i) The projection map $V \twoheadrightarrow V_J$ induces a surjective affine N_J-equivariant map $p_J \colon \mathscr{A} \twoheadrightarrow \mathscr{A}_J$ sending φ_0 to $\varphi_{0,J}$.

(ii) Taking inverse images the map p_J induces N_J-equivariant injections $\Phi_J^{\mathrm{aff}} \hookrightarrow \Phi^{\mathrm{aff}}$ and $\mathfrak{H}_J \hookrightarrow \mathfrak{H}$.

(iii) We have naturally $W_J^{\mathrm{aff}} \subseteq W^{\mathrm{aff}}$ and the inclusion $\Sigma_J^{\mathrm{aff}} \subseteq \Sigma^{\mathrm{aff}}$ is N_J-equivariant.

Proof. (i) It is clear that $p_J \colon \mathscr{A} \to \mathscr{A}_J$ is a surjective affine map. We check that p_J is N_J-equivariant. Let $n \in N_J$, $v \in V$, $\alpha \in \Phi_J$, and $u \in U_\alpha$ be arbitrary. Denote by $\overline{v}$ the image of v in V_J and write $w := {}^{\mathrm{v}}\nu(n) \in W_{0,J}$. Recalling Lemma 1.15 we compute

$$\begin{aligned}\big(n.(\varphi_0 + v)\big)_\alpha(u) &= (\varphi_0 + v)_{w^{-1}(\alpha)}(n^{-1}un)\\ &= (\varphi_0)_{w^{-1}(\alpha)}(n^{-1}un) + \langle w^{-1}(\alpha), v\rangle\\ &= (\varphi_{0,J})_{w^{-1}(\alpha)}(n^{-1}un) + \langle w^{-1}(\alpha), \overline{v}\rangle\\ &= \big(n.(\varphi_{0,J} + \overline{v})\big)_\alpha(u).\end{aligned}$$

For $n = 1$ this computation shows that $p_J(\varphi)$ is the restriction of the valuation $\varphi \in \mathscr{A}$ to the root group datum $\big(Z, (U_\alpha)_{\alpha\in\Phi_J}\big)$. This observation and the computation for general $n \in N_J$ imply the N_J-equivariance of p_J.

(ii) Given $\alpha \in \Phi_J$, we have $\langle \alpha, x - \varphi_0\rangle = \langle \alpha, p_J(x) - \varphi_{0,J}\rangle$ for all $x \in \mathscr{A}$. Therefore, we have

$$\begin{aligned}p_J^{-1}(a_{\alpha,r}) &= p_J^{-1}\left(\{y \in \mathscr{A}_J \,|\, \langle \alpha, y - \varphi_{0,J}\rangle + r \geq 0\}\right)\\ &= \{x \in \mathscr{A} \,|\, \langle \alpha, x - \varphi_0\rangle + r \geq 0\}\end{aligned}$$

for all $\alpha \in \Phi_J$ and $r \in \Gamma_\alpha$. Therefore, as p_J is surjective and N_J-equivariant, we have an N_J-equivariant inclusion $\Phi_J^{\mathrm{aff}} \hookrightarrow \Phi^{\mathrm{aff}}$. The same computation with "$\geq$" replaced by "$=$" shows that we have an N_J-equivariant inclusion $\mathfrak{H}_J \hookrightarrow \mathfrak{H}$.

(iii) The construction in Proposition 1.32 and (ii) imply $\Sigma_J = V_J^* \cap \Sigma$. Hence, we have an inclusion $\Sigma_J^{\mathrm{aff}} = \Sigma_J \times \mathbb{Z} \subseteq \Sigma^{\mathrm{aff}} = \Sigma \times \mathbb{Z}$ which is evidently N_J-equivariant (1.5.3). Recall that W_J is contained in W. Under this inclusion we have $S(\mathfrak{H}_J) \subseteq S(\mathfrak{H})$ and therefore $W_J^{\mathrm{aff}} \subseteq W^{\mathrm{aff}}$. □

Remark 1.57. With the notations from above we have $p_J(\mathfrak{C}) \subseteq \mathfrak{C}_J$, but in general this need not be an equality. In this case, we have $S_J^{\mathrm{aff}} \not\subseteq S^{\mathrm{aff}}$. In particular, the length and Bruhat order on W_J^{aff} (resp. W_J, resp. $W_J(1)$) (see Lemma 1.47) is not the one induced by the length and Bruhat order of W^{aff} (resp. W, resp. $W(1)$). Nevertheless, we have $S_J \subseteq S$ and hence the length (resp. Bruhat order) of W_0 induces the length (resp. Bruhat order) of $W_{0,J}$.

Lemma 1.58 (Iwahori decomposition). *Denote by $I_{M_J}(1)$ the pro-p radical of the parahoric group $I_{M_J} := K_{\mathfrak{C}_J}$. Write $I_{U_J} := I(1) \cap U_J = I \cap U_J$ and $I_{U_J^{\mathrm{op}}} := I(1) \cap U_J^{\mathrm{op}} = I \cap U_J^{\mathrm{op}}$. We have $I_{M_J}(1) = I(1) \cap M_J$ and a decomposition (with respect to any order of the three factors)*

$$I(1) = I_{U_J^{\mathrm{op}}} I_{M_J}(1) I_{U_J}. \tag{1.8.1}$$

Proof. This is a direct consequence of (1.6.2) applied to $I(1)$ and to $I_{M_J}(1)$: the multiplication map induces homeomorphisms

$$\prod_{\alpha\in\Sigma^-} U_{(\alpha,1)} \times Z_0(1) \times \prod_{\alpha\in\Sigma^+} U_{(\alpha,0)} \cong I(1) \text{ and } \prod_{\alpha\in\Sigma_J^-} U_{(\alpha,1)} \times Z_0(1) \times \prod_{\alpha\in\Sigma_J^+} U_{(\alpha,0)} \cong I_{M_J}(1).$$

The result follows from $\prod_{\alpha\in\Sigma^-\smallsetminus\Sigma_J} U_{(\alpha,1)} \cong I_{U_J^{\mathrm{op}}}$ and $\prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_J} U_{(\alpha,0)} \cong I_{U_J}$. □

Remark 1.59. By [OV18, 2.1.2] (see also [Vig16, (53)]) the multiplication map induces homeomorphisms

$$\prod_{\alpha\in\Sigma^+} U_{(\alpha,0)} \cong I(1)\cap U^+ = K\cap U^+ \quad \text{and} \quad \prod_{\alpha\in\Sigma^-} U_{(\alpha,1)} \cong I(1)\cap U^- = K(1)\cap U^-.$$

Definition 1.60. (a) An element $m \in M_J$ is called *M_J-positive* (or *positive* if no confusion arises) if it satisfies

$$mI_{U_J}m^{-1} \subseteq I_{U_J} \quad \text{and} \quad I_{U_J^{\mathrm{op}}} \subseteq mI_{U_J^{\mathrm{op}}}m^{-1}. \tag{1.8.2}$$

We denote by M_J^+ (or $M_J^{+,G}$ if we want to emphasize that M_J is considered as a Levi subgroup in G) the submonoid of M_J-positive elements. As $I_{M_J}(1)$ normalizes both I_{U_J} and $I_{U_J^{\mathrm{op}}}$, we have $I_{M_J}(1)M_J^+I_{M_J}(1) \subseteq M_J^+$. The elements in $M_J^- := M_J^{-,G} := \left\{m \in M_J \,\middle|\, m^{-1} \in M_J^+\right\}$ are called *M_J-negative* (or *negative*).

We have $I_{U_J^{\mathrm{op}}} = K_{U_J^{\mathrm{op}}}(1) := K(1)\cap U_J^{\mathrm{op}}$ and $I_{U_J} = K_{U_J} := K\cap U_J$. Therefore, $K_J := K_{\{\varphi_{0,J}\}} = K\cap M_J$ (this equality is justified in [Vig, Prop. 4.2]) normalizes both $I_{U_J^{\mathrm{op}}}$ and I_{U_J}, and hence we have $K_J \subseteq M_J^+\cap M_J^-$.

(b) An element $m \in M_J^+$ is called *strictly M_J-positive in G* (or *strictly positive* if no confusion arises) if it satisfies the following conditions:

- ◇ m is contained in the center of M_J;
- ◇ for all compact open subgroups U_1, U_2 of U_J there exists $n \in \mathbb{N}$ such that $m^nU_1m^{-n} \subseteq U_2$;
- ◇ for all compact open subgroups U_1, U_2 of U_J^{op} there exists $n \in \mathbb{N}$ such that $U_1 \subseteq m^nU_2m^{-n}$.

Notice that a central element $m \in M_J^+$ is strictly M_J-positive if and only if for some (hence any) compact open subgroups $U_1 \subseteq U_J$ and $U_2 \subseteq U_J^{\mathrm{op}}$ we have $\bigcap_{n\in\mathbb{N}} m^nU_1m^{-n} = \{1\}$ and $\bigcap_{n\in\mathbb{N}} m^{-n}U_2m^n = \{1\}$, because each element in U_J (resp. U_J^{op}) is contained in a compact open subgroup of U_J (resp. U_J^{op}).

An element $m \in M$ is called *strictly M_J-negative* (or strictly negative) if m^{-1} is strictly M_J-positive.

(c) An element $w \in W_J$ is called *M_J-positive* (or *positive* if no confusion arises) if $w(\Sigma^+\smallsetminus\Sigma_J) \subseteq \Sigma^{\mathrm{aff},+}$. We denote by $W_{M_J^+}$ (or $W_{M_J^{+,G}}$ when the surrounding group is not clear from the context) the set of positive elements and call the elements in $W_{M_J^-} := W_{M_J^{-,G}} := (W_{M_J^+})^{-1}$ *M_J-negative* (or *negative*).

Remark 1.61. There exist strictly M_J-positive elements in M_J [BK98, (6.14) Lem.]. Their usefulness stems from the following property: given a strictly M_J-positive element $a \in M_J$ and any $m \in M_J$, there exists $n \in \mathbb{N}$ such that $a^nm \in M_J^+$.

Proposition 1.62. *(i) We have $W_{M_J^+} \cong \Lambda_{M_J^+} \rtimes W_{0,J}$, where*

$$\Lambda_{M_J^+} := \Lambda_{M_J^{+,G}} := \left\{\lambda \in \Lambda \,\middle|\, \langle\alpha, \nu(\lambda)\rangle \le 0 \quad \textit{for all } \alpha \in \Sigma^+\smallsetminus\Sigma_J\right\}. \tag{1.8.3}$$

In particular, $W_{M_J^+}$ *is a monoid.*

(ii) *The Bruhat decompositions of* M_J *induce bijections* $W_{M_J^+} \cong I_{M_J}\backslash M_J^+/I_{M_J}$ *and* $W_{M_J^+}(1) \cong I_{M_J}(1)\backslash M_J^+/I_{M_J}(1)$.

(iii) *Let* $m \in M_J$ *be central and let* $\lambda \in \Lambda$, $w_0 \in W_{0,J}$ *such that* $e^\lambda w_0$ *represents* $I_{M_J} m I_{M_J}$. *Then* m *is strictly* M_J*-positive if and only if* $w_0 = 1$ *and*

$$\lambda \in \left\{\lambda \in \Lambda \;\middle|\; \begin{array}{l}\langle \alpha, \nu(\lambda)\rangle < 0 \;\textit{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_J \textit{ and}\\ \langle \alpha, \nu(\lambda)\rangle = 0 \;\textit{ for all } \alpha \in \Sigma_J\end{array}\right\} \subseteq \Lambda_{M_J^+}.$$

In this case we also call λ strictly M_J-positive *(or* strictly positive*).*

Proof. (i) See [Vig15, § 2.1]. Let $w = e^\lambda w_0 \in W_J$ for $\lambda \in \Lambda$ and $w_0 \in W_{0,J}$.

Claim 1. *The element* w_0 *permutes* $\Sigma^+ \smallsetminus \Sigma_J$.

Proof of the claim. We can write $w_0 = s_1 \cdots s_n$ with $s_1, \dots, s_n \in S_J$, so we may suppose $w_0 = s_{\alpha,\alpha^\vee} \in S_J$ with $\alpha \in \Pi_J$. It is well-known that $w_0\big(\Sigma^+ \smallsetminus \{\alpha\}\big) = \Sigma^+ \smallsetminus \{\alpha\}$. Since $\alpha \in \Sigma_J$ and $w_0(\Sigma_J) = \Sigma_J$, this proves the claim. □

Claim 2. *We have* $w \in W_{M_J^+}$ *if and only if* $\lambda \in \Lambda_{M_J^+}$.

Proof of the claim. Recall that $e^\lambda w_0(\alpha, 0) = \big(w_0(\alpha), -\langle w_0(\alpha), \nu(\lambda)\rangle\big)$ for each $\alpha \in \Sigma$ (1.5.3). By Claim 1 we have $w_0(\Sigma^+ \smallsetminus \Sigma_J) = \Sigma^+ \smallsetminus \Sigma_J$. Hence, we have an equivalence

$$w = e^\lambda w_0 \in W_{M_J^+} \iff -\langle \alpha, \nu(\lambda)\rangle \geq 0 \text{ for all } \alpha \in w_0(\Sigma^+ \smallsetminus \Sigma_J) = \Sigma^+ \smallsetminus \Sigma_J.$$

□

Claim 3. *The group* $W_{0,J}$ *normalizes* $\Lambda_{M_J^+}$.

Proof of the claim. Using (1.6.6) we have $\langle w_0(\alpha), \nu(\lambda)\rangle = \langle \alpha, w_0^{-1}.\nu(\lambda)\rangle = \langle \alpha, \nu(w_0^{-1} e^\lambda w_0)\rangle$. Hence, the claim follows from Claim 1. □

Claims 2 and 3 together imply $W_{M_J^+} = \Lambda_{M_J^+} \rtimes W_{0,J}$.

(ii) See [OV18, Rem. 2.11 (2)]. By Proposition 1.46 we have a bijection $W_J \cong I_{M_J}\backslash M_J/I_{M_J}$. Let $m \in M_J$ and take $w = e^\lambda w_0 \in W_J$, with $\lambda \in \Lambda$ and $w_0 \in W_{0,J}$, representing $I_{M_J} m I_{M_J}$. Since $I_{M_J} \subseteq M^+$ we may assume $m \in N_J = N \cap M_J$. It follows from Remark 1.59 that the multiplication map induces homeomorphisms $\prod_{\alpha \in \Sigma^+ \smallsetminus \Sigma_J} U_{(\alpha,0)} \cong I_{U_J}$ and $\prod_{\alpha \in \Sigma^- \smallsetminus \Sigma_J} U_{(\alpha,1)} \cong I_{U_J^{\mathrm{op}}}$. Since $m U_{(\alpha,k)} m^{-1} = U_{(w_0(\alpha), k - \langle w_0(\alpha), \nu(\lambda)\rangle)}$ and hence also $m^{-1} U_{(\alpha,k)} m = U_{(w_0^{-1}(\alpha), k + \langle \alpha, \nu(\lambda)\rangle)}$ for all $(\alpha, k) \in \Sigma^{\mathrm{aff}}$, we have the equivalences

$$\begin{aligned} m I_{U_J} m^{-1} \subseteq I_{U_J} &\iff m U_{(\alpha,0)} m^{-1} \subseteq \prod_{\beta \in \Sigma^+ \smallsetminus \Sigma_J} U_{(\beta,0)} \text{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_J \\ &\iff -\langle w_0(\alpha), \nu(\lambda)\rangle \geq 0 \text{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_J \\ &\iff \lambda \in \Lambda_{M^+} \iff w \in W_{M_J^+} \\ &\iff -\langle -\alpha, \nu(\lambda)\rangle \geq 0 \text{ for all } \alpha \in \Sigma^- \smallsetminus \Sigma_J \\ &\iff \langle \alpha, \nu(\lambda)\rangle \geq 0 \text{ for all } \alpha \in \Sigma^- \smallsetminus \Sigma_J \\ &\iff m^{-1} U_{(\alpha,1)} m \subseteq \prod_{\beta \in \Sigma^- \smallsetminus \Sigma_J} U_{(\beta,1)} \text{ for all } \alpha \in \Sigma^- \smallsetminus \Sigma_J \\ &\iff m^{-1} I_{U_J^{\mathrm{op}}} m \subseteq I_{U_J^{\mathrm{op}}}. \end{aligned}$$

In particular, m is positive if and only if $w \in W_{M_J^+}$. A similar argument shows $W_{M_J^+}(1) \cong I_{M_J}(1)\backslash M^+/I_{M_J}(1)$.

(iii) Let $m \in M_J$ be central and let $w = e^\lambda w_0 \in \Lambda \rtimes W_{0,J}$ representing $I_{M_J} m I_{M_J}$. Since m is central, we have $m \in Z$. This implies $w_0 = 1$, i. e. $w = \lambda \in \Lambda$. For each $\alpha \in \Sigma_J$ we have $U_{(\alpha,0)} = mU_{(\alpha,0)}m^{-1} = U_{(\alpha,-\langle\alpha,\nu(\lambda)\rangle)}$, i. e. $\langle\alpha,\nu(\lambda)\rangle = 0$. For all $\alpha \in \Sigma \smallsetminus \Sigma_J$, $k \in \mathbb{Z}$, and $n \in \mathbb{N}$ we have $m^n U_{(\alpha,k)} m^{-n} = U_{(\alpha,k-n\cdot\langle\alpha,\nu(\lambda)\rangle)}$.

Consider the compact open subgroups $I_{U_J} = \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_J} U_{(\alpha,0)}$ of U_J and $K_{U_J^{\mathrm{op}}} := \prod_{\alpha\in\Sigma^-\smallsetminus\Sigma_J} U_{(\alpha,0)}$ of U_J^{op}. We then obtain the equivalences

$$\begin{aligned}
\bigcap_{n\in\mathbb{N}} m^n I_{U_J} m^{-n} = \{1\} &\iff \langle\alpha,\nu(\lambda)\rangle < 0 \text{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_J \\
&\iff \langle\alpha,\nu(\lambda)\rangle > 0 \text{ for all } \alpha \in \Sigma^- \smallsetminus \Sigma_J \\
&\iff \bigcap_{n\in\mathbb{N}} m^{-n} K_{U_J^{\mathrm{op}}} m^n = \{1\}.
\end{aligned}$$

The statement follows. □

Corollary 1.63. *An element $m \in M_J$ is positive if and only if $mI_{U_J}m^{-1} \subseteq I_{U_J}$.*

Proof. See the chain of equivalences in the proof of Proposition 1.62 (ii). □

2. Generalities on Hecke algebras

In this section we collect the necessary preliminaries pertaining to abstract Hecke rings in general, and pro-p Iwahori-Hecke algebras in particular. Nothing in this section is original work.

2.1. Abstract Hecke rings

We give a short introduction to abstract Hecke rings. The standard reference is the monograph [AZ95, Ch. 3, §1].

Let G be a group and $\Gamma \subseteq G$ a subgroup such that the set $\Gamma\backslash\Gamma g\Gamma$ of right cosets in $\Gamma g\Gamma$ is finite for each $g \in G$.

Lemma 2.1. *Let $g \in G$ and put $\Gamma_{(g)} := \Gamma \cap g^{-1}\Gamma g$. Given a decomposition*

$$\Gamma = \bigsqcup_{\gamma_i \in \Gamma_{(g)}\backslash\Gamma} \Gamma_{(g)}\gamma_i,$$

where "$\gamma_i \in \Gamma_{(g)}\backslash\Gamma$" means that γ_i runs through a set of representatives of the right cosets in $\Gamma_{(g)}\backslash\Gamma$, we have

$$\Gamma g\Gamma = \bigsqcup_{\gamma_i \in \Gamma_{(g)}\backslash\Gamma} \Gamma g\gamma_i.$$

In particular, the number $\mu(g) := \mu_\Gamma(g) := |\Gamma\backslash\Gamma g\Gamma|$ equals the index $[\Gamma : \Gamma_{(g)}]$ of $\Gamma_{(g)}$ inside Γ.

Proof. See [AZ95, Ch. 3, Lem. 1.2]. □

Definition 2.2. Two subgroups Γ_1 and Γ_2 of G are called *commensurable*, if the indices $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ are both finite. In this case we write $\Gamma_1 \sim \Gamma_2$.

Lemma 2.3. *Commensurability is an equivalence relation. The set*

$$\widetilde{\Gamma} := \{g \in G \,|\, g^{-1}\Gamma g \sim \Gamma\}$$

is a group. It is called the commensurator *of Γ in G. The elements of $\widetilde{\Gamma}$ are called Γ-rational.*

Proof. See [AZ95, Ch. 3, Lem. 1.3 and Lem. 1.4]. □

Remark 2.4. (a) If $N := N_G(\Gamma)$ denotes the normalizer of Γ inside G, the map $\mu_\Gamma \colon G \to \mathbb{N}$ is constant on NgN for all $g \in G$. Given $g \in G$ and $x, y \in N$, we have

$$\begin{aligned}\Gamma_{(xgy)} &= \Gamma \cap (y^{-1}g^{-1}x^{-1})\Gamma(xgy) \\ &= y^{-1} \cdot \big(y\Gamma y^{-1} \cap g^{-1}(x^{-1}\Gamma x)g\big) \cdot y = y^{-1}\Gamma_{(g)}y.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_\Gamma(xgy) &= \big[\Gamma : \Gamma_{(xgy)}\big] = \big[\Gamma : y^{-1}\Gamma_{(g)}y\big] \\ &= \big[y\Gamma y^{-1} : \Gamma_{(g)}\big] = \big[\Gamma : \Gamma_{(g)}\big] = \mu_\Gamma(g).\end{aligned}$$

(b) The map $\delta_G \colon G \to \mathbb{Q}^\times$, $g \mapsto \mu_\Gamma(g) \cdot \mu_\Gamma(g^{-1})^{-1}$ is a group homomorphism. To see this, notice that $\Gamma_{(g^{-1})} = g\Gamma_{(g)}g^{-1}$ and hence

$$\delta_G(g) = \frac{\mu_\Gamma(g)}{\mu_\Gamma(g^{-1})} = \frac{[\Gamma : \Gamma_{(g)}]}{[\Gamma : \Gamma_{(g^{-1})}]} = \frac{[\Gamma : \Gamma_{(g)}]}{[g^{-1}\Gamma g : \Gamma_{(g)}]} = [\Gamma : g^{-1}\Gamma g],$$

where $[\Gamma : g^{-1}\Gamma g]$ denotes the generalized index. Given $g, h \in G$, we therefore compute

$$\begin{aligned}\delta_G(gh) &= [\Gamma : h^{-1}g^{-1}\Gamma gh] = [\Gamma : h^{-1}\Gamma h] \cdot [h^{-1}\Gamma h : h^{-1}g^{-1}\Gamma gh] \\ &= [\Gamma : h^{-1}\Gamma h] \cdot [\Gamma : g^{-1}\Gamma g] = \delta_G(h) \cdot \delta_G(g).\end{aligned}$$

(c) Let $\Gamma' \subseteq G$ be another subgroup that is commensurable with Γ. Then we have $[\Gamma : g^{-1}\Gamma g] = [\Gamma' : g^{-1}\Gamma' g]$ for all $g \in G$. It clearly suffices to show this in the case $\Gamma' \subseteq \Gamma$. From $[\Gamma : \Gamma'] = [g^{-1}\Gamma g : g^{-1}\Gamma' g]$ it follows that

$$[\Gamma : g^{-1}\Gamma g] = \frac{[\Gamma : \Gamma'] \cdot [\Gamma' : g^{-1}\Gamma' g]}{[g^{-1}\Gamma g : g^{-1}\Gamma' g]} = [\Gamma' : g^{-1}\Gamma' g].$$

Definition 2.5. Let $S \subseteq G$ be a subset closed under multiplication. The tuple (Γ, S) is called a *Hecke pair* if $\Gamma \subseteq S \subseteq \widetilde{\Gamma}$.

We fix a commutative ring A with 1. Let (Γ, S) be a Hecke pair and consider the free A-module $X_A(\Gamma, S)$ on the basis $\{(\Gamma g) \,|\, \Gamma g \in \Gamma\backslash S\}$. Via

$$(\Gamma g) \cdot \gamma := (\Gamma g \gamma), \quad g, \gamma \in S \tag{2.1.1}$$

the semigroup S acts on $X_A(\Gamma, S)$ by A-linear transformations.

Consider the A-submodule $H_A(\Gamma, S)$ of $X_A(\Gamma, S)$ given by

$$H_A(\Gamma, S) := X_A(\Gamma, S)^\Gamma := \{t \in X_A(\Gamma, S) \,|\, t\gamma = t \text{ for all } \gamma \in \Gamma\}.$$

Lemma 2.6. *Given elements $t := \sum_i a_i \cdot (\Gamma g_i)$ of $H_A(\Gamma, S)$ and $t' := \sum_j b_j \cdot (\Gamma h_j)$ of $X_A(\Gamma, S)$, the element*

$$t \cdot t' := \sum_{i,j} a_i b_j \cdot (\Gamma g_i h_j) \in X_A(\Gamma, S)$$

does not depend on the choice of representatives and belongs to $H_A(\Gamma, S)$ whenever $t' \in H_A(\Gamma, S)$.

In this way $H_A(\Gamma, S)$ becomes an associative A-algebra with unit (Γ), and $X_A(\Gamma, S)$ is a left module over $H_A(\Gamma, S)$, called the universal module.

Proof. See the discussion in [AZ95, Ch. 3, §1.2] following Lemma 1.4. □

Definition 2.7. We call $H_A(\Gamma, S)$ the *Hecke ring* (or Hecke algebra) over A attached to the Hecke pair (Γ, S).

Lemma 2.8. *Given $g \in S$, we define*

$$(g) := (g)_\Gamma := (\Gamma g \Gamma) := \sum_{g_i \in \Gamma \backslash \Gamma g \Gamma} (\Gamma g_i) \in X_A(\Gamma, S).$$

Then $H_A(\Gamma, S)$ is a free A-module with basis $\{(g) \mid g \in S\}$.

The multiplication on $H_A(\Gamma, S)$ can be described as follows: let $g, g' \in S$ and write

$$\Gamma g \Gamma = \bigsqcup_{i=1}^{\mu(g)} \Gamma g_i, \qquad \Gamma g' \Gamma = \bigsqcup_{j=1}^{\mu(g')} \Gamma g'_j.$$

Then we have

$$(g) \cdot (g') = \sum_{\Gamma h \Gamma \subseteq \Gamma g \Gamma g' \Gamma} c(g, g'; h) \cdot (h),$$

where $c(g, g'; h) := \left|\left\{(g_i, g'_j) \,\middle|\, g_i g'_j \in \Gamma h\right\}\right|$. Moreover, we have

$$c(g, g'; h) = v(g, g'; h) \cdot \mu(g') \mu(h)^{-1},$$

where $v(g, g'; h) := \left|\{g_i \mid g_i g' \in \Gamma h \Gamma\}\right|$.

Proof. See [AZ95, Ch. 3, Lem. 1.5]. □

Proposition 2.9. *Let (Γ, S) and (Γ_0, S_0) be two Hecke pairs satisfying*

$$\Gamma_0 \subseteq \Gamma, \quad S \subseteq \Gamma S_0, \quad \text{and} \quad \Gamma \cap S_0 \cdot S_0^{-1} \subseteq \Gamma_0. \tag{2.1.2}$$

Then

$$\varepsilon \colon H_A(\Gamma, S) \hookrightarrow H_A(\Gamma_0, S_0), \quad \sum_i a_i \cdot (\Gamma g_i) \longmapsto \sum_i a_i \cdot (\Gamma_0 g_i),$$

where we may choose $g_i \in S_0$, is a well-defined injective homomorphism of A-algebras.

Proof. See [AZ95, Ch. 3, Prop. 1.9]. The idea is to define an injective A-linear map

$$X_A(\Gamma, S) \to X_A(\Gamma_0, S_0), \quad (\Gamma g) \mapsto (\Gamma_0 g_0),$$

where g_0 is a coset representative of Γg in S_0; this is possible since $S \subseteq \Gamma S_0$. The map is well-defined because of $\Gamma \cap S_0 \cdot S_0^{-1} \subseteq \Gamma_0$, and it is injective because of $\Gamma_0 \subseteq \Gamma$. These last two conditions also ensure that Γ-invariant elements are mapped into Γ_0-invariant elements. □

Proposition 2.10. *Let (Γ, S) be a Hecke pair for the group G. Then (Γ, S^{-1}) is a Hecke pair and the map*

$$j \colon H_A(\Gamma, S) \longrightarrow H_A(\Gamma, S^{-1}), \qquad (g)_\Gamma \longmapsto (g^{-1})_\Gamma \tag{2.1.3}$$

is an anti-isomorphism of A-algebras. In particular, if S is a group, then j is an anti-automorphism on $H_A(\Gamma, S)$.

Proof. See [AZ95, Ch. 3, Prop. 1.11]. □

Lemma 2.11. *Let (Γ, S) and (Γ_0, S_0) be two Hecke pairs satisfying (2.1.2), and suppose that the Hecke pairs (Γ, S^{-1}) and (Γ_0, S_0^{-1}) also satisfy (2.1.2). Then we have a commutative diagram*

$$\begin{tikzcd} H_A(\Gamma, S) \arrow[r, "\varepsilon"] \arrow[d, "j"'] & H_A(\Gamma_0, S_0) \arrow[d, "j"] \\ H_A(\Gamma, S^{-1}) \arrow[r, "\varepsilon"'] & H_A(\Gamma_0, S_0^{-1}). \end{tikzcd}$$

Proof. See [AZ95, Ch. 3, Lem. 1.13]. □

Lemma 2.12. *Let (Γ, S) be a Hecke pair for the group G. Let $\delta\colon S \to (A, \cdot)$ be a monoid homomorphism with $\Gamma \subseteq \mathrm{Ker}(\delta)$. Then the A-linear map $H_A(\Gamma, S) \to H_A(\Gamma, S)$ given by $(g)_\Gamma \mapsto \delta(g) \cdot (g)_\Gamma$ is a homomorphism of A-algebras.*

Proof. Let $g, g' \in S$. For any $h \in S$ with $\Gamma h \Gamma \subseteq \Gamma g \Gamma g' \Gamma$ we have $\delta(h) = \delta(g) \cdot \delta(g')$ because of $\Gamma \subseteq \mathrm{Ker}(\delta)$. With Lemma 2.8 we have

$$\big(\delta(g) \cdot (g)_\Gamma\big) \cdot \big(\delta(g') \cdot (g')_\Gamma\big) = \sum_{\Gamma h \Gamma \subseteq \Gamma g \Gamma g' \Gamma} c(g, g'; h) \cdot \big(\delta(h) \cdot (h)_\Gamma\big).$$

Hence, the map $H_A(\Gamma, S) \mapsto H_A(\Gamma, S)$, $(g)_\Gamma \mapsto \delta(g) \cdot (g)_\Gamma$ preserves the product on $H_A(\Gamma, S)$. □

2.2. The pro-p Iwahori-Hecke algebra

The most important example of a Hecke ring in view of the representation theory of a connected reductive group with coefficients in a field of characteristic p is the pro-p Iwahori-Hecke algebra studied by Vignéras [Vig16].

As in section 1 we fix a local field F with finite residue field κ_F of characteristic $p > 0$ and cardinality q.

Let $\mathbf{G}$ be a connected reductive group over F, $\mathbf{T}$ a maximal F-split torus with centralizer $\mathbf{Z} := \mathbf{C_G}(\mathbf{T})$ and normalizer $\mathbf{N} := \mathbf{N_G}(\mathbf{T})$. Recall the relative root system $\Phi := \Phi(\mathbf{G}, \mathbf{T})$, the apartment $\mathscr{A}$ (1.4.1) with its set of hyperplanes $\mathfrak{H}$ (1.4.3), the chosen fundamental alcove $\mathfrak{C}$, and the special point $\varphi_0 \in \overline{\mathfrak{C}}$. Let Δ be the basis of Φ such that the corresponding set of positive roots Φ^+ equals $\{\alpha \in \Phi \mid \langle \alpha, x - \varphi_0 \rangle > 0\}$, where $x \in \mathfrak{C}$ is some arbitrary point (see the end of Remark 1.2 (c)). Let $I \subseteq G$ be the Iwahori subgroup corresponding to $\mathfrak{C}$, and $I(1)$ its pro-p radical (see Definition 1.42). The Iwahori-Weyl group $W := N/Z_0$ (resp. the pro-p Iwahori-Weyl group $W(1) := N/Z_0(1)$) parametrizes the double cosets in $I \backslash G / I$ (resp. $I(1) \backslash G / I(1)$) (see Proposition 1.46) and acts on $\mathscr{A}$ and $\mathfrak{H}$ via the natural action $\nu\colon N \to \mathrm{Aut}\,\mathscr{A}$ (see Proposition 1.27).

We fix a commutative ring R with 1.

Definition 2.13. The tuple $\big(I(1),G\big)$ is a Hecke pair since $I(1)$ is a compact open subgroup of G. The corresponding Hecke algebra $\mathcal{H}_R(G) := H_R\big(I(1),G\big)$ with coefficients in R is called the *pro-p Iwahori-Hecke algebra*. By construction we have

$$\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G). \tag{2.2.1}$$

We write $T_w := T_g := \big(I(1)gI(1)\big)$ for the element $w \in W(1)$ representing the double coset $I(1)gI(1)$. Then $\mathcal{H}_R(G)$ is the free R-module with basis $\{T_w \mid w \in W(1)\}$. We refer to this basis as the *Iwahori-Matsumoto basis* of $\mathcal{H}_R(G)$.

Let $s \in S^{\mathrm{aff}}$. Let $\alpha \in \Phi^+_{\mathrm{red}}$ and $r \in \Gamma_\alpha$ be such that $H_s = H_{\alpha,r}$. Put $\alpha_s := \alpha$ and $r_s := r$ if $r \in \Gamma'_\alpha$, but $\alpha_s := 2\alpha$ and $r_s := 2r$ if $r \notin \Gamma'_\alpha$ (in which case $2r \in \Gamma_{2\alpha}$). For each $\tilde{s} \in S^{\mathrm{aff}}(1)$ lifting s we define

$$q_{\tilde{s}} := q_s := |U_{\alpha_s,r_s}/U_{\alpha_s,r_s+}|. \tag{2.2.2}$$

The numbers q_s are powers of $q = |\kappa_F|$. They are used to describe the relations that determine $\mathcal{H}_R(G)$. Choose some $u \in U_{\alpha_s,r_s} \smallsetminus U_{\alpha_s,r_s+}$ and let $m_s(u)$ be the unique element in $N \cap U_{-\alpha_s} u U_{-\alpha_s}$ (see Remark 1.4 (b)). Denote by $s(u)$ the image of $m_s(u)$ in $W(1)$. It lifts s (see the discussion following [Vig16, Prop. 4.1]) and is called an *admissible lift*.

Recall the (possibly non-split) exact sequence

$$1 \longrightarrow Z_\kappa \longrightarrow W(1) \longrightarrow W \longrightarrow 1,$$

where Z_κ is a finite abelian subgroup of $W(1)$. We write $w \bullet t := wtw^{-1} \in Z_\kappa$ for $w \in W(1)$ and $t \in Z_\kappa$. This extends to an action of $W(1)$ on the group algebra $R[Z_\kappa]$.

Theorem 2.14. *[Vig16, Thm. 2.2] The pro-p Iwahori-Hecke algebra $\mathcal{H}_R(G)$ is a free R-module on the basis $\{T_w \mid w \in W(1)\}$ and satisfies the following relations, which determine its R-algebra structure:*

(1) Braid relations: $T_w T_{w'} = T_{ww'}$ for all $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$;
(2) Quadratic relations: $T_{s(u)}^2 = q_s T_{s(u)^2} + c_{s(u)} T_{s(u)}$ for $s \in S^{\mathrm{aff}}$, for certain $c_{s(u)} \in R[Z_\kappa]$.[5]

Remark 2.15. (a) Let $s \in S^{\mathrm{aff}}$, $s(u) \in S^{\mathrm{aff}}(1)$ as above, and $\tilde{s} \in S^{\mathrm{aff}}(1)$ be an arbitrary lift of s. Then we have $\tilde{s} = s(u) \cdot t$ for some $t \in Z_\kappa$ and the quadratic relation

$$T_{\tilde{s}}^2 = q_s T_{\tilde{s}^2} + c_{\tilde{s}} T_{\tilde{s}},$$

where $c_{\tilde{s}} := c_{s(u)} \cdot T_t$ [Vig16, Lem. 4.6]. This can be seen directly using the braid and quadratic relations, and that Z_κ is normal in $W(1)$:

$$\begin{aligned} T_{\tilde{s}}^2 &= T_{s(u)} T_t T_{s(u)} T_t \\ &= T_{s(u)}^2 \cdot T_{s(u)^{-1} t s(u) t} \\ &= \big(q_s T_{s(u)^2} + c_{s(u)} T_{s(u)}\big) \cdot T_{s(u)^{-1} t s(u) t} \\ &= q_s T_{s(u)^2} T_{s(u)^{-1} t s(u) t} + c_{s(u)} T_{s(u)} T_{s(u)^{-1} t s(u) t} \\ &= q_s T_{(s(u)t)^2} + c_{s(u)} T_t \cdot T_{s(u)t} \\ &= q_s T_{\tilde{s}^2} + c_{\tilde{s}} T_{\tilde{s}}. \end{aligned}$$

[5]As a consequence of the braid relations the group algebra $R[Z_\kappa]$ embeds into $\mathcal{H}_R(G)$ via $t \mapsto T_t$ ($t \in Z_\kappa$).

(b) Since $\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G)$ we actually have $c_s \in \mathbb{Z}[Z_\kappa]$ for each $s \in S^{\mathrm{aff}}(1)$. For all $s \in S^{\mathrm{aff}}(1)$, $t \in Z_\kappa$, and $w \in W(1)$ with $wsw^{-1} \in S^{\mathrm{aff}}(1)$ we have (see [Vig16, (67)])

$$q_s = q_{st} = q_{wsw^{-1}}, \qquad c_{st} = c_s \cdot t, \quad \text{and} \quad w \bullet c_s = c_{wsw^{-1}}. \tag{2.2.3}$$

The equalities $q_s = q_{st}$ and $c_{st} = c_s \cdot t$ are true by definition. Write $z := wsw^{-1}$ and compare coefficients on both sides of $T_z(T_wT_s) = (T_zT_w)T_s$. Notice that $zw = ws$. Consider two cases: If $\ell(zw) = \ell(ws) = \ell(w) + 1$, then $\ell(z^{-1}ws) = \ell(w) < \ell(z(z^{-1}ws))$ as well as $\ell(zws^{-1}) = \ell(w) < \ell((zws^{-1})s)$ and hence, by the braid relations,

$$\begin{aligned}
T_z(T_wT_s) &= T_zT_{zz^{-1}ws} = T_z^2T_{z^{-1}ws} \\
&= (q_zT_{z^2} + c_zT_z)T_{z^{-1}ws} = q_zT_{zws} + c_zT_{ws}, \quad \text{and} \\
(T_zT_w)T_s &= T_{zws^{-1}s}T_s = T_{zws^{-1}}T_s^2 \\
&= T_{zws^{-1}}(q_sT_{s^2} + c_sT_s) = q_sT_{zws} + (zws^{-1}) \bullet c_s \cdot T_{zw}.
\end{aligned}$$

Comparing coefficients and using $zws^{-1} = w$, we obtain $q_s = q_z = q_{wsw^{-1}}$ and $w \bullet c_s = (zws^{-1}) \bullet c_s = c_z = c_{wsw^{-1}}$.

If on the other hand $\ell(zw) = \ell(ws) = \ell(w) - 1$, we have $\ell(z(z^{-1}w)) = \ell(w) > \ell(z^{-1}w)$ and $\ell((ws^{-1})s) = \ell(w) > \ell(ws^{-1})$. Hence, we compute

$$\begin{aligned}
T_zT_w &= T_z^2T_{z^{-1}w} = (q_zT_{z^2} + c_zT_z)T_{z^{-1}w} = q_zT_{zw} + c_zT_w, \quad \text{and} \\
T_wT_s &= T_{ws^{-1}}T_s^2 = T_{ws^{-1}} \cdot (q_sT_{s^2} + c_sT_s) = q_sT_{ws} + (ws^{-1}) \bullet c_s \cdot T_w.
\end{aligned}$$

Using these relations, as well as $\ell(zws) > \ell(ws)$ and $\ell(zws) > \ell(zw)$, we obtain

$$\begin{aligned}
(T_zT_w)T_s &= q_zT_{zw}T_s + c_zT_wT_s = q_zT_{zws} + q_sc_zT_{ws} + c_z \cdot (ws^{-1}) \bullet c_s \cdot T_w, \\
T_z(T_wT_s) &= q_sT_zT_{ws} + (zws^{-1}) \bullet c_s \cdot T_zT_w \\
&= q_sT_{zws} + q_z \cdot (zws^{-1}) \bullet c_sT_{zw} + (zws^{-1}) \bullet c_s \cdot c_z \cdot T_w
\end{aligned}$$

Again, comparing coefficients gives $q_{wsw^{-1}} = q_z = q_s$ and $q_z \cdot (zws^{-1}) \bullet c_s = q_s \cdot c_z = q_z \cdot c_{wsw^{-1}}$. Dividing by q_z (in $\mathbb{Z}[Z_\kappa]$) yields $w \bullet c_s = (zws^{-1}) \bullet c_s = c_{wsw^{-1}}$. All in all, this shows (2.2.3).

Given any $s \in S^{\mathrm{aff}}(1)$, the quadratic relations imply that T_s is invertible in $\mathcal{H}_R(G)$ if and only if q_s is invertible in R. This motivates the following definition of T_s^*, which is of importance even when q_s is not invertible.

Lemma 2.16. *Let $s \in S^{\mathrm{aff}}(1)$ and define $T_s^* := T_s - c_s$. Then we have*

$$T_s^*T_s = T_sT_s^* = q_sT_{s^2} \quad \textit{and} \quad T_{s^{-1}}^*T_s = T_sT_{s^{-1}}^* = q_s.$$

For $u \in \Omega(1)$ we have $c_{u^{-1}su} = T_u^{-1}c_sT_u$ and $T_{u^{-1}su}^* = T_u^{-1}T_s^*T_u$.

Proof. See [Vig16, Lem. 4.12]. By (2.2.3) we have $s \bullet c_s = c_s$ and hence $T_s c_s = s \bullet c_s T_s = c_s T_s$. Similarly, $s \bullet c_{s^{-1}} = c_{s^{-1}}$ implies $T_s c_{s^{-1}} = c_{s^{-1}} T_s$. Therefore, $T_s^* T_s = T_s T_s^*$ and $T_{s^{-1}}^* T_s = T_s T_{s^{-1}}^*$. Further, we have $T_s^* T_s = (T_s - c_s)T_s = T_s^2 - c_s T_s = q_s T_{s^2}$ and

$$\begin{aligned} T_{s^{-1}}^* T_s &= (T_{s^{-1}} - c_{s^{-1}})T_s = T_{s^{-1}} T_s - c_{s^{-1}} T_s = T_{s^{-2}} T_s^2 - c_{s^{-1}} T_s \\ &= q_s T_{s^{-2}} T_{s^2} + T_{s^{-2}} c_s T_s - c_{s^{-1}} T_s = q_s + c_{s^{-1}} T_s - c_{s^{-1}} T_s = q_s. \end{aligned}$$

Given $u \in \Omega(1)$, we have $T_u^{-1} c_s T_u = u^{-1} \bullet c_s T_u^{-1} T_u = c_{u^{-1}su}$ and then also

$$T_{u^{-1}su}^* = T_{u^{-1}su} - c_{u^{-1}su} = T_u^{-1} T_s T_u - T_u^{-1} c_s T_u = T_u^{-1}(T_s - c_s)T_u = T_u^{-1} T_s^* T_u. \quad \square$$

Proposition 2.17. *Let $w \in W(1)$, write $w = s_1 \cdots s_{\ell(w)} u$ with $s_1, \ldots, s_{\ell(w)} \in S^{\mathrm{aff}}(1)$ and $u \in \Omega(1)$, and define*

$$T_{w^{-1}}^* := T_u^{-1} T_{s_{\ell(w)}^{-1}}^* \cdots T_{s_1^{-1}}^*. \tag{2.2.4}$$

The following properties hold:

(i) T_w is invertible in $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(G)$ with inverse $q_w^{-1} T_{w^{-1}}^*$, where $q_w := q_{s_1} \cdots q_{s_{\ell(w)}}$.

(ii) We have $T_{w^{-1}}^* T_w = T_w T_{w^{-1}}^* = q_w$ in $\mathcal{H}_R(G)$.

(iii) We have $T_w^* = T_{s_1}^* \cdots T_{s_{\ell(w)}}^* T_u$. This decomposition and the definition of q_w do not depend on the decomposition of w.

(iv) We have $T_w^* = T_w + \sum_{v<w} \lambda_v T_v$ in $\mathcal{H}_R(G)$ for certain $\lambda_v \in \mathbb{Z}$ (where "$<$" denotes the Bruhat order in $W(1)$).

Proof. See [Vig16, Prop. 4.13]. By the braid relations, $T_w = T_{s_1} \cdots T_{s_{\ell(w)}}$ does not depend on the decomposition of w, and from Lemma 2.16 it follows that $T_w T_{w^{-1}}^* = T_{w^{-1}}^* T_w = q_w$ in $\mathcal{H}_R(G)$. Therefore, T_w is invertible in $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(G)$ with inverse $q_w^{-1} T_{w^{-1}}^*$. As inverses are unique, it follows that $T_{w^{-1}}^*$ and q_w do not depend on the decomposition of w. Since $T_{w^{-1}}^*$ actually lies in $\mathcal{H}_{\mathbb{Z}}(G) \subseteq \mathcal{H}_{\mathbb{Z}[p^{-1}]}(G)$, this is also true in $\mathcal{H}_{\mathbb{Z}}(G)$, and hence in $\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G)$.

We replace w by w^{-1}. Then T_w^* does not depend on the decomposition

$$w^{-1} = u^{-1} s_{\ell(w)}^{-1} \cdots s_1^{-1} = (u^{-1} s_{\ell(w)} u) \cdots (u^{-1} s_1^{-1} u) \cdot u^{-1}$$

of w^{-1}. By Lemma 2.16, we compute

$$\begin{aligned} T_w^* &= T_{u^{-1}}^{-1} T_{u^{-1}s_1 u}^* \cdots T_{u^{-1}s_{\ell(w)}u}^* \\ &= T_u \cdot (T_u^{-1} T_{s_1}^* T_u) \cdots (T_u^{-1} T_{s_{\ell(w)}}^* T_u) = T_{s_1}^* \cdots T_{s_{\ell(w)}}^* T_u. \end{aligned}$$

Finally, (iv) holds in $\mathcal{H}_{\mathbb{Z}}(G)$, as follows from multiplying out $(T_{s_1} - c_{s_1}) \cdots (T_{s_{\ell(w)}} - c_{s_{\ell(w)}})T_u$ and from the definition of the Bruhat order (Lemma 1.47, (ii), use Proposition 1.37, (iii)). As $\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G)$, the statement follows. $\square$

Remark 2.18. Given $w \in W(1)$ and $u \in \Omega(1)$, we have $q_{wu} = q_w$ as follows from the definition of q_w. If $\overline{w} \in W^{\mathrm{aff}}$ and $u \in \Omega$ are such that $\overline{w}u$ is the image of w in W, we may thus define $q_{\overline{w}} := q_{\overline{w}u} := q_w$; if, moreover, $\overline{w} = s_1 \cdots s_{\ell(\overline{w})}$ is a decomposition with $s_1, \ldots, s_{\ell(\overline{w})} \in S^{\mathrm{aff}}$, then

$$q_{\overline{w}u} = q_{\overline{w}} = q_{s_1} \cdots q_{s_{\ell(\overline{w})}}. \tag{2.2.5}$$

Lemma 2.19. *For all $v, w \in W(1)$ there exists a unique positive integer $q_{v,w} \in q^{\mathbb{Z}_{\geq 0}}$ such that*

$$q_v q_w = q_{vw} q_{v,w}^2. \tag{2.2.6}$$

For each $t, t' \in Z_\kappa$ we have $q_{vt,wt'} = q_{v,w}$. It thus makes sense to define $q_{\overline{v},\overline{w}} := q_{v,w}$, where $\overline{v}$ (resp. $\overline{w}$) denotes the image of v (resp. w) in W.

Proof. See [Vig16, Lem. 4.19] or [Vig06, Lem. 1.2, 1)]. The uniqueness and the last statement are clear. By the above remark we may assume $v \in W$ and $w \in W^{\mathrm{aff}}$. We prove (2.2.6) by induction on $\ell(w)$. If $\ell(w) = 0$ there is nothing to show. Assume that (2.2.6) holds for $v \in W$ and $w \in W^{\mathrm{aff}}$. Let $s \in S^{\mathrm{aff}}$ such that $\ell(ws) = \ell(w) + 1$. If $\ell(vws) = \ell(vw) + 1$, then

$$q_v q_{ws} = q_v q_w q_s = q_{vw} q_s q_{v,w}^2 = q_{vws} q_{v,w}^2,$$

so (2.2.6) holds with $q_{v,ws} := q_{v,w}$. Assume now that $\ell(vws) = \ell(vw) - 1$. Let $s_1, \ldots, s_{\ell(vw)} \in S^{\mathrm{aff}}$ and $u \in \Omega$ with $vw = s_1 \cdots s_{\ell(vw)} u$. By the deletion condition (see Proposition 1.36, (i)) and the definition of $\ell(vw)$ there exists $1 \leq i \leq \ell(vw)$ such that $s_1 \cdots s_{\ell(vw)}(usu^{-1}) = s_1 \cdots \widehat{s_i} \cdots s_{\ell(vw)}$. Rearranging gives $(s_{i+1} \cdots s_{\ell(vw)} u) s (s_{i+1} \cdots s_{\ell(vw)} u)^{-1} = s_i$, i. e. s and s_i are conjugate in W. From (2.2.3) it follows that $q_s = q_{s_i}$. Thus, we compute

$$q_v q_{ws} = q_v q_w q_s = q_{vw} q_s q_{v,w}^2 = q_{s_1} \cdots \widehat{q_{s_i}} \cdots q_{s_{\ell(vw)}} \cdot q_{s_i} q_s q_{v,w}^2 = q_{vws} \cdot (q_s q_{v,w})^2,$$

and the statement follows with $q_{v,ws} := q_{v,w} q_s$. □

Definition 2.20. Let o be an orientation of $(\mathscr{A}, \mathfrak{H})$ (Definition 1.52) and recall the map $\varepsilon_o \colon W(1) \times S^{\mathrm{aff}}(1) \to \{\pm 1\}$ (1.7.1). For each $(w, s) \in W(1) \times S^{\mathrm{aff}}(1)$ we define

$$T_s^{\varepsilon_o(w,s)} := \begin{cases} T_s, & \text{if } \varepsilon_o(w,s) = 1; \\ T_s^* = T_s - c_s, & \text{if } \varepsilon_o(w,s) = -1. \end{cases}$$

This is an abuse of notation, since $T_s^{\varepsilon_o(w,s)} \neq T_s^{-1}$ whenever $\varepsilon_o(w,s) = -1$. Yet, it is unlikely that this will be a cause of confusion. Given $s_1, \ldots, s_n \in S^{\mathrm{aff}}(1)$ and $u, u' \in \Omega(1)$, we define

$$E_o(u, s_1, \ldots, s_n, u') = T_u T_{s_1}^{\varepsilon_o(u,s_1)} \cdots T_{s_i}^{\varepsilon_o(us_1\cdots s_{i-1}, s_i)} \cdots T_{s_n}^{\varepsilon_o(us_1\cdots s_{n-1}, s_n)} T_{u'}.$$

Theorem 2.21. *Let o be an orientation of $(\mathscr{A}, \mathfrak{H})$, let $w, w' \in W(1)$, and write $w = s_1 \cdots s_{\ell(w)} u$ for $s_1, \ldots, s_{\ell(w)} \in S^{\mathrm{aff}}(1)$ and $u \in \Omega$. Then the element*

$$E_o(w) := E_o(1, s_1, \ldots, s_{\ell(w)}, u) \in \mathcal{H}_R(G)$$

only depends on w, and we have $E_o(w) E_{o \bullet w}(w') = q_{w,w'} E_o(ww')$.

Proof. See [Vig16, Thm. 5.25]. □

Remark 2.22. Let o be an orientation of $(\mathcal{A}, \mathfrak{H})$ and $w \in W(1)$. Then we have

$$E_o(w) = T_w + \sum_{v<w} \lambda_v T_v \in \mathcal{H}_R(G), \quad \text{for some } \lambda_v \in \mathbb{Z} \tag{2.2.7}$$

(with the same proof as in Proposition 2.17 (iv)). Therefore, $\{E_o(w) \mid w \in W(1)\}$ is a basis of $\mathcal{H}_R(G)$, called an *alcove walk basis*. It was first described and studied in Schmidt's Diplomarbeit [Sch09]; see also his PhD thesis [Sch19].

Corollary 2.23 (Fundamental Lemma)**.** *Let $v, w \in W(1)$. In $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(G)$ we have*

$$q_{v,w} T_v^{-1} T_{vw} = T_w + \sum_{w'<w} \lambda_{w'} T_{w'} \in \mathcal{H}_{\mathbb{Z}}(G), \quad \textit{for some } \lambda_{w'} \in \mathbb{Z}.$$

Proof. Let o be the trivial orientation of $(\mathcal{A}, \mathfrak{H})$ (Example 1.55), i. e. the one oriented away from the fundamental alcove. Then $E_o(w) = T_w$ for all $w \in W(1)$. Given $v, w \in W(1)$, we have

$$T_v E_{o\bullet v}(w) = E_o(v) E_{o\bullet v}(w) = q_{v,w} E_o(vw) = q_{v,w} T_{vw}$$

by Theorem 2.21. Multiplying T_v^{-1} from the left, the statement follows from (2.2.7). □

Theorem 2.21 and Corollary 2.23 highlight the important role of the integers $q_{v,w}$. We end this section by describing a different view on the $q_{v,w}$, which is due to Vignéras [Vig16, Sec. 4.4].

Definition 2.24. For each $w \in W^{\mathrm{aff}}$ and each $s \in S^{\mathrm{aff}}$ we define $q(wH_s) := q_s \in q^{\mathbb{N}}$.

As W^{aff} acts simply transitively on the set of alcoves of $\mathcal{A}$ (Proposition 1.36, (ii)), $\{H_s \mid s \in S^{\mathrm{aff}}\}$ is the set of walls of $\mathfrak{C}$, and each hyperplane is a wall of some alcove, this defines an integer $q(H)$ for each $H \in \mathfrak{H}$.

This definition is unambiguous by (2.2.3): given $w, w' \in W^{\mathrm{aff}}$ and $s, s' \in S^{\mathrm{aff}}$ with $wH_s = w'H_{s'}$ we have $H_s = w^{-1}w'H_{s'} = H_{w^{-1}w'sw'^{-1}w}$ and thus $w^{-1}w's'w'^{-1}w = s \in S^{\mathrm{aff}}$. Therefore, $q(wH_s) = q_s = q_{s'} = q(w'H_{s'})$.

Lemma 2.25. *Let $w \in W^{\mathrm{aff}}$. Recall the set $\mathfrak{H}_w$ of hyperplanes separating $\mathfrak{C}$ and $w\mathfrak{C}$ (Proposition 1.36, (iv)). We have $q_w = \prod_{H \in \mathfrak{H}_w} q(H)$.*

Proof. Let $w = s_1 \cdots s_{\ell(w)}$ be a decomposition with $s_1, \dots, s_{\ell(w)} \in S^{\mathrm{aff}}$. By Proposition 1.36, (iv) we have $\mathfrak{H}_w = \{H_{s_1}, s_1 H_{s_2}, (s_1 s_2) H_{s_3}, \dots, (s_1 \cdots s_{\ell(w)-1}) H_{s_{\ell(w)}}\}$. We deduce

$$q_w = q_{s_1} \cdots q_{s_{\ell(w)}} = q(H_{s_1}) \cdot q(s_1 H_{s_2}) \cdots q\big((s_1 \cdots s_{\ell(w)-1}) H_{s_{\ell(w)}}\big) = \prod_{H \in \mathfrak{H}_w} q(H). \quad \square$$

Lemma 2.26. *Let $v, w \in W^{\mathrm{aff}}$ and $u, u' \in \Omega$.*

(i) We have $\mathfrak{H}_{vw} = \big(\mathfrak{H}_v \cup v\mathfrak{H}_w\big) \smallsetminus \big(\mathfrak{H}_v \cap v\mathfrak{H}_w\big)$.
(ii) We have $q_{v,w} = \prod_{H \in \mathfrak{H}_v \cap v\mathfrak{H}_w} q(H)$.
(iii) We have $q_{vu,wu'} = q_{v,uwu^{-1}}$ and in particular $q_{vu,wu'} = \prod_{H \in \mathfrak{H}_{vu} \cap vu\mathfrak{H}_{wu'}} q(H)$.

Proof. See [Vig16, Lem. 4.17, Lem. 4.19].

(i) Let $\Gamma_v := (\mathfrak{D}_0, \dots, \mathfrak{D}_{\ell(v)})$ be a minimal gallery connecting $\mathfrak{D}_0 = \mathfrak{C}$ and $\mathfrak{D}_{\ell(v)} = v\mathfrak{C}$, and let $\Gamma_w := (\mathfrak{D}'_0, \dots, \mathfrak{D}'_{\ell(w)})$ be a minimal gallery connecting $\mathfrak{D}'_0 = \mathfrak{C}$ and $\mathfrak{D}'_{\ell(w)} = w\mathfrak{C}$. Then Γ_v (resp. Γ_w) crosses the hyperplanes in $\mathfrak{H}_v$ (resp. $\mathfrak{H}_w$) exactly once, and only those. The compound gallery

$$\Gamma := (\mathfrak{D}_0, \dots, \mathfrak{D}_{\ell(v)}, v\mathfrak{D}'_0, \dots, v\mathfrak{D}'_{\ell(w)})$$

connects $\mathfrak{C}$ and $vw\mathfrak{C}$. By construction, it crosses the hyperplanes in $\mathfrak{H}_v \cup v\mathfrak{H}_w$, and only those. Also, Γ crosses the hyperplanes in $\mathfrak{H}_v \cap v\mathfrak{H}_w$ (and only those) twice. Hence, the set of hyperplanes crossed by Γ exactly once (equivalently here: an odd number of times) is $(\mathfrak{H}_v \cup v\mathfrak{H}_w) \smallsetminus (\mathfrak{H}_v \cap v\mathfrak{H}_w)$, which thus coincides with $\mathfrak{H}_{vw}$.

(ii) By (i) and Lemma 2.25 we have

$$\begin{aligned} q_v q_w &= \prod_{H \in \mathfrak{H}_v} q(H) \cdot \prod_{H \in \mathfrak{H}_w} q(H) = \prod_{H \in \mathfrak{H}_v} q(H) \cdot \prod_{H \in v\mathfrak{H}_w} q(H) \\ &= \prod_{H \in (\mathfrak{H}_v \cup v\mathfrak{H}_w) \smallsetminus (\mathfrak{H}_v \cap v\mathfrak{H}_w)} q(H) \cdot \prod_{H \in \mathfrak{H}_v \cap v\mathfrak{H}_w} q(H)^2 \\ &= q_{vw} \cdot \prod_{H \in \mathfrak{H}_v \cap v\mathfrak{H}_w} q(H)^2. \end{aligned}$$

The definition of $q_{v,w}$ implies $q_{v,w} = \prod_{H \in \mathfrak{H}_v \cap v\mathfrak{H}_w} q(H)$.

(iii) By definition we have $q_{vu} = q_v$ and $q_{wu'} = q_w$. Using (2.2.3) we see $q_{uwu^{-1}} = q_w$. Together we have

$$q_{vuwu^{-1}} q^2_{v,uwu^{-1}} = q_v q_{uwu^{-1}} = q_{vu} q_{wu'} = q_{vuwu'} q^2_{vu,wu'} = q_{vuwu^{-1}} q^2_{vu,wu'}.$$

The claim follows from the uniqueness assertion in Lemma 2.19. The last statement now follows using $\mathfrak{H}_{vu} = \mathfrak{H}_v$, $\mathfrak{H}_{wu'} = \mathfrak{H}_w$ and $\mathfrak{H}_{uwu^{-1}} = u\mathfrak{H}_w$. □

2.3. The positive subalgebra

Retain the notations of section 2.2. Recall the pro-p Iwahori-Hecke algebra $\mathcal{H}_R(G) = H_R(I(1), G)$, with coefficients in the commutative ring R with 1, attached to the F-points $G = \mathbf{G}(F)$ of a connected reductive group $\mathbf{G}$ over F.

We choose a (standard) Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ as in section 1.8 associated with a subset J of the basis Δ of $\Phi = \Phi(\mathbf{G}, \mathbf{T})$. Then $\mathbf{M}$ is again a connected reductive group over F. We write $\Phi_M := \Phi(\mathbf{M}, \mathbf{T})$. Let $\Psi := \Phi^+ \smallsetminus \Phi_M$ and write Ψ_{red} for the reduced roots in Ψ. Denote by $\mathbf{P}$ (resp. $\mathbf{P}^{\mathrm{op}}$) the (standard) parabolic subgroup containing $\mathbf{M}$ with unipotent radical $\mathbf{U}_P := \prod_{\alpha \in \Psi_{\mathrm{red}}} \mathbf{U}_\alpha$ (resp. $\mathbf{U}_{P^{\mathrm{op}}} := \prod_{\alpha \in -\Psi_{\mathrm{red}}} \mathbf{U}_\alpha$); it satisfies $P = M \ltimes U_P$ (resp. $P^{\mathrm{op}} = M \ltimes U_{P^{\mathrm{op}}}$). Let $I_M = I \cap M$ be the Iwahori subgroup and $I_M(1) = I(1) \cap M$ the pro-p Iwahori subgroup of M. Write $I_{U_{P^{\mathrm{op}}}} := I \cap U_{P^{\mathrm{op}}} = I(1) \cap U_{P^{\mathrm{op}}}$ and $I_{U_P} := I \cap U_P = I(1) \cap U_P$. By Lemma 1.58 we have a decomposition

$$I(1) = I_{U_{P^{\mathrm{op}}}} I_M(1) I_{U_P}$$

with respect to any order of the three factors. Recall the monoid M^+ consisting of the elements $m \in M$ with $m I_{U_P} m^{-1} \subseteq I_{U_P}$ and $I_{U_{P^{\mathrm{op}}}} \subseteq m I_{U_{P^{\mathrm{op}}}} m^{-1}$.

We denote by $W_{0,M}$ (resp. W_M, resp. $W_M(1)$) the finite (resp. Iwahori-, resp. pro-p Iwahori-) Weyl group of M.

To M is attached the pro-p Iwahori-Hecke algebra $\mathcal{H}_R(M)$. Given $m \in M$ and $w \in W_M(1)$ representing $I_M(1)mI_M(1)$, we write $T_w^M := T_m^M := \big(I_M(1)mI_M(1)\big)$. Then $\big\{T_w^M \,\big|\, w \in W_M(1)\big\}$ is the Iwahori-Matsumoto basis of $\mathcal{H}_R(M)$.

Definition 2.27. We call $\mathcal{H}_R(M^+) := H_R\big(I_M(1), M^+\big)$ the *positive subalgebra* of $\mathcal{H}_R(M)$. By Proposition 1.62, (ii) the set $\big\{T_w^M \,\big|\, w \in W_{M^+}(1)\big\}$ is a basis of $\mathcal{H}_R(M^+)$.

Proposition 2.28. *Consider the injective R-linear map*

$$\theta\colon \mathcal{H}_R(M) \longrightarrow \mathcal{H}_R(G), \quad T_m^M \longmapsto T_m. \tag{2.3.1}$$

The restriction $\theta^+ := \theta\big|_{\mathcal{H}_R(M^+)}$ respects the product.

Proof. See [Vig98, II.5 Prop.]. □

Proposition 2.29. *The map $\theta^+\colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ extends to a homomorphism of R-algebras $\widetilde{\theta}^+\colon \mathcal{H}_R(M) \to \mathcal{H}_R(G)$ if and only if there exists a strictly positive element $a \in M$ such that T_a is invertible in $\mathcal{H}_R(G)$. In this case, T_a is invertible for all strictly positive elements $a \in M$, and $\widetilde{\theta}^+$ is unique.*

Proof. See [Vig98, II.6 Prop.]. Let $Z(M)$ be the center of M. The map $Z(M) \to \mathcal{H}_R(M)^\times$, $m \mapsto T_m^M$ is a group homomorphism because of $I_M(1)mI_M(1) = I_M(1)m$ and the definition of the product in $\mathcal{H}_R(M)$. By Remark 1.61 there exists a strictly positive element $a \in M$ and that $\mathcal{H}_R(M)$ is the localization of $\mathcal{H}_R(M^+)$ at the central element T_a^M. The proposition is a formal consequence of this fact.

Assume that $\widetilde{\theta}^+$ exists. Let $a \in Z(M)$ be a strictly positive element. By the above remark the element T_a^M is invertible in $\mathcal{H}_R(M)$, and hence $T_a = \widetilde{\theta}^+(T_a^M)$ is invertible in $\mathcal{H}_R(G)$. Given any $m \in M$, there exists $n \in \mathbb{N}$ such that $a^n m \in M^+$. We compute

$$\begin{aligned}\theta^+\big(T_{a^n m}^M\big) &= \widetilde{\theta}^+\big(T_{a^n}^M T_m^M\big) = \widetilde{\theta}^+\big(T_{a^n}^M\big) \cdot \widetilde{\theta}^+\big(T_m^M\big) \\ &= \theta^+\big((T_a^M)^n\big) \cdot \widetilde{\theta}^+\big(T_m^M\big) = T_a^n \cdot \widetilde{\theta}^+\big(T_m^M\big).\end{aligned}$$

Rearranging yields

$$\widetilde{\theta}^+\big(T_m^M\big) = T_a^{-n} \cdot \theta^+\big(T_{a^n m}^M\big). \tag{2.3.2}$$

This shows that $\widetilde{\theta}^+$ is unique.

Conversely, assume that there exists a strictly positive element $a \in Z(M)$ such that T_a is invertible in $\mathcal{H}_R(G)$. We define a map $\widetilde{\theta}^+\colon \mathcal{H}_R(M) \to \mathcal{H}_R(G)$ by (2.3.2) and R-linear extension. We first check well-definedness, i. e. independence of the choice of n. Take any $m \in M$ and let $n, r \in \mathbb{N}$ such that $a^n m \in M^+$ and $a^r m \in M^+$. We may assume $r \geq n$. We compute

$$\begin{aligned}T_a^{-r} \cdot \theta^+\big(T_{a^r m}^M\big) &= T_a^{-r} \cdot \theta^+\big(T_{a^{r-n}}^M \cdot T_{a^n m}^M\big) = T_a^{-r} \cdot \theta^+\big((T_a^M)^{r-n}\big) \cdot \theta^+\big(T_{a^n m}^M\big) \\ &= T_a^{-r} \cdot T_a^{r-n} \cdot \theta^+\big(T_{a^n m}^M\big) = T_a^{-n} \cdot \theta^+\big(T_{a^n m}^M\big).\end{aligned}$$

Therefore, $\widetilde{\theta}^+$ is well-defined. It remains to prove that $\widetilde{\theta}^+$ respects the product. Take any $m, m' \in M$. There exist $m_1, \ldots, m_r \in M$ such that

$$I_M(1)mI_M(1)m'I_M(1) = \bigsqcup_{i=1}^{r} I_M(1)m_iI_M(1). \tag{2.3.3}$$

By Lemma 2.8 we have $T_m^M \cdot T_{m'}^M = \sum_{i=1}^r c(m, m'; m_i) \cdot T_{m_i}^M$ for certain $c(m, m'; m_i) \in R$. Take $n, n' \in \mathbb{N}$ such that $a^n m$ and $a^{n'} m'$ lie in M^+. Multiplying (2.3.3) by $a^{n+n'}$ on both sides and using $I_M(1) \subseteq M^+$, we observe that $a^{n+n'} m_i \in M^+$ for all $1 \le i \le r$. We compute

$$\begin{aligned}
\widetilde{\theta}^+(T_m^M \cdot T_{m'}^M) &= \widetilde{\theta}^+\left(\sum_{i=1}^r c(m, m'; m_i) \cdot T_{m_i}^M\right) = \sum_{i=1}^r c(m, m'; m_i) \cdot \widetilde{\theta}^+(T_{m_i}^M) \\
&= \sum_{i=1}^r c(m, m'; m_i) \cdot T_a^{-(n+n')} \cdot \theta^+(T_{a^{n+n'} m_i}^M) \\
&= T_a^{-(n+n')} \cdot \theta^+\left(T_{a^{n+n'}}^M \cdot \left(\sum_{i=1}^r c(m, m'; m_i) \cdot T_{m_i}^M\right)\right) \\
&= T_a^{-(n+n')} \cdot \theta^+(T_{a^{n+n'}}^M \cdot T_m^M \cdot T_{m'}^M) = T_a^{-n} \cdot T_a^{-n'} \cdot \theta^+(T_{a^n m}^M \cdot T_{a^{n'} m'}^M) \\
&= T_a^{-n} \cdot \theta^+(T_{a^n m}^M) \cdot T_a^{-n'} \cdot \theta^+(T_{a^{n'} m'}^M) = \widetilde{\theta}^+(T_m^M) \cdot \widetilde{\theta}^+(T_{m'}^M).
\end{aligned}$$

Hence, $\widetilde{\theta}^+$ is a homomorphism of R-algebras. This finishes the proof. □

3. Parabolic induction for the general linear group

We first point our attention to the general linear group. In this case, some proofs simplify and the only prerequisite is section 2.1 (Abstract Hecke rings). In particular, this section does not rely on section 1 (Review of Bruhat-Tits theory).

Let F be a local field with normalized valuation $\omega\colon F \to \mathbb{Z} \cup \{\infty\}$. We denote by $\mathcal{O}_F$, $\mathfrak{m}_F$, κ_F the valuation ring, its maximal ideal and the residue field, respectively. We fix a uniformizer π_F of F. The residue field κ_F is a finite field of order q and characteristic p.

We fix a commutative ring R with 1.

Let $n \in \mathbb{N}$ and consider the general linear group $G := \mathrm{GL}_n(F)$ of invertible $n \times n$-matrices with entries in F. We fix the maximal compact subgroup $K := \mathrm{GL}_n(\mathcal{O}_F)$. Let $\mathbf{B}_n(\kappa_F)$ be the subgroup of $\mathrm{GL}_n(\kappa_F)$ consisting of upper triangular matrices, and let $\mathbf{U}_n(\kappa_F)$ be the subgroup of upper triangular matrices with 1's on the diagonal. The Iwahori subgroup I (resp. pro-p Iwahori subgroup $I(1)$) of G is given by the preimage of $\mathbf{B}_n(\kappa_F)$ (resp. $\mathbf{U}_n(\kappa_F)$) under the mod-$\mathfrak{m}_F$ projection map $\mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\kappa_F)$.

We fix a standard parabolic subgroup

$$P := P_{n_1,\dots,n_r} := \begin{pmatrix} \mathrm{GL}_{n_1}(F) & * & * & * \\ & \mathrm{GL}_{n_2}(F) & * & * \\ & & \ddots & * \\ & & & \mathrm{GL}_{n_r}(F) \end{pmatrix} \subseteq G$$

with standard Levi subgroup M and unipotent radical U_P given, respectively, by

$$M = \begin{pmatrix} \mathrm{GL}_{n_1}(F) & & & \\ & \mathrm{GL}_{n_2}(F) & & \\ & & \ddots & \\ & & & \mathrm{GL}_{n_r}(F) \end{pmatrix}, \qquad U_P = \begin{pmatrix} E_{n_1} & * & * & * \\ & E_{n_2} & * & * \\ & & \ddots & * \\ & & & E_{n_r} \end{pmatrix},$$

where E_k denotes the $k \times k$ identity matrix. We have a semidirect product $P = M \ltimes U_P$.

We will restrict our attention to the pro-p Iwahori subgroup $I(1)$ and remark that the following results also hold mutatis mutandis for the Iwahori subgroup I.

Notation 3.1. We denote by $\mathrm{pr}_M\colon P \twoheadrightarrow M$ the canonical projection map with kernel U_P. Given $g \in P$, we write $g_M := \mathrm{pr}_M(g) \in M$ and $g_U := g_M^{-1} g \in U_P$. These are the unique elements in M and U_P, respectively, with $g = g_M g_U$.

The subgroups $I_P(1) := I(1) \cap P$, $I_M(1) := I(1) \cap M$, and $I_{U_P} := I(1) \cap U_P$ are compact open in P, M and U_P, respectively. We have a semidirect product

$$I_P(1) = I_M(1) \ltimes I_{U_P}.$$

In particular, given $g \in I_P(1)$, we have $g_M \in I_M(1)$ and $g_U \in I_{U_P}$.

We denote by $P^{\mathrm{op}} = {}^{\mathrm{t}}P$ the opposite parabolic subgroup and by $U_{P^{\mathrm{op}}} = {}^{\mathrm{t}}U_P$ its unipotent radical. Then $I_{P^{\mathrm{op}}}(1) := I(1) \cap P^{\mathrm{op}}$ and $I_{U_{P^{\mathrm{op}}}} := I(1) \cap U_{P^{\mathrm{op}}}$ are compact open in P^{op} and $U_{P^{\mathrm{op}}}$, respectively. We have a semidirect product

$$I_{P^{\mathrm{op}}}(1) = I_M(1) \ltimes I_{U_{P^{\mathrm{op}}}}.$$

We consider the Hecke algebras $H_R\big(I_P(1), P\big)$, $H_R\big(I_M(1), M\big)$ and $H_R\big(I(1), G\big)$ (see section 2.1). Our goal is to describe two homomorphisms of R-algebras

$$\begin{tikzcd} & H_R\big(I_P(1),P\big) \arrow[dl, "\widetilde{\Theta}_G^P"'] \arrow[dr, "\Theta_M^P"] & \\ H_R\big(I(1),G\big) & & H_R\big(I_M(1),M\big). \end{tikzcd}$$

Given these morphisms, it is possible to describe the *parabolic induction* functor from the category of $H_R\big(I_M(1), M\big)$-modules to the category of $H_R\big(I(1), G\big)$-modules. More precisely, given a right $H_R\big(I_M(1), M\big)$-module $\mathcal{M}$, it becomes a right $H_R\big(I_P(1), P\big)$-module by putting

$$m \cdot x := m \cdot \Theta_M^P(x) \quad \text{for } x \in H_R\big(I_P(1), P\big) \text{ and } m \in \mathcal{M}.$$

Now, we obtain a right $H_R\big(I(1), G\big)$-module

$$\mathcal{M} \otimes_{H_R(I_P(1),P)} H_R\big(I(1), G\big).$$

We say it is *parabolically induced* from $\mathcal{M}$. The functor thus described coincides with the parabolic induction functor already studied in the literature (e. g. [Vig15] and [OV18]).

3.1. Positive elements and the positive subalgebra

Definition 3.2. An element $m \in M$ is called *positive* if it satisfies

$$m I_{U_P} m^{-1} \subseteq I_{U_P} \quad \text{and} \quad I_{U_{P^{\mathrm{op}}}} \subseteq m I_{U_{P^{\mathrm{op}}}} m^{-1}.$$

We denote by M^+ the monoid of positive elements and by $M^- := (M^+)^{-1}$ the monoid of *negative* elements. As $K_M := K \cap M$ normalizes both $I_{U_P} = K \cap U_P$ and $I_{U_{P^{\mathrm{op}}}} = K \cap U_{P^{\mathrm{op}}}$, we have $K_M \subseteq M^+ \cap M^-$.

Notice that the *Cartan decomposition* for G holds, i. e.

$$G = \bigsqcup_{\substack{d_1,\dots,d_n \in \mathbb{Z} \\ d_1 \geq d_2 \geq \cdots \geq d_n}} K \operatorname{diag}\big(\pi_F^{d_1}, \dots, \pi_F^{d_n}\big) K. \tag{3.1.1}$$

Given $A \in K \operatorname{diag}\big(\pi_F^{d_1}, \dots, \pi_F^{d_n}\big) K$ with $d_1 \geq d_2 \geq \cdots \geq d_n$, we write $d_j(A) := d_j$ for $1 \leq j \leq n$.

The next lemma characterizes M^+.

Lemma 3.3. *We have*

$$\begin{aligned} M^+ &= \left\{ \mathrm{diag}(A_1, \dots, A_r) \in M \,\middle|\, A_i \in \mathrm{GL}_{n_i}(F) \textit{ and } d_{n_i}(A_i) \geq d_1(A_{i+1}) \textit{ for all } i \right\} \\ &= \left\{ m \in M \,\middle|\, m I_{U_P} m^{-1} \subseteq I_{U_P} \right\}. \end{aligned}$$

Proof. Consider an element $A = \mathrm{diag}(A_1, \dots, A_r) \in M$ with diagonal matrices $A_i = \mathrm{diag}\big(\pi_F^{d_1(A_i)}, \dots, \pi_F^{d_{n_i}(A_i)}\big)$ for each $1 \leq i \leq r$. For all $1 \leq i < j \leq r$ we have

$$\begin{aligned} A_i \,\mathrm{Mat}_{n_i,n_j}(\mathcal{O}_F) A_j^{-1} \subseteq \mathrm{Mat}_{n_i,n_j}(\mathcal{O}_F) &\iff d_s(A_i) \geq d_t(A_j) \quad \text{for all } s, t \\ &\iff d_{n_i}(A_i) \geq d_1(A_j) \\ &\iff \mathrm{Mat}_{n_j,n_i}(\mathcal{O}_F) \subseteq A_j \,\mathrm{Mat}_{n_j,n_i}(\mathcal{O}_F) A_i^{-1}. \end{aligned}$$

It follows that A lies in M^+ if and only if it lies in any of the other two sets. Notice that all the sets in question are unions of double cosets with respect to K_M. The assertion now follows from the Cartan decomposition for M. □

Notation 3.4. Let G be a group and H a subgroup of G. Given $g, h \in G$, we use the following notation:

$$h^g := g^{-1} h g, \quad H^g := \left\{ h^g \,\middle|\, h \in H \right\}, \quad \text{and} \quad H_{(g)} := H^g \cap H.$$

The following proposition is a generalization of [Gri88, Lem. 2]. We remark here, that the next result remains true if we replace $I(1)$ by any compact open subgroup Γ of G satisfying $\Gamma \cap P = (\Gamma \cap M)(\Gamma \cap U_P)$; this includes K and I.

Proposition 3.5. *Let $g \in P$ be arbitrary. The indices of $(I_M(1))_{(g_M)}$ in $I_M(1)$, of $\mathrm{pr}_M\big((I_P(1))_{(g)}\big)$ in $(I_M(1))_{(g_M)}$, and of $\big(I_{U_P}\big)_{(g)}$ in I_{U_P} are finite. If we write*

$$I_M(1) = \bigsqcup_{i=1}^{\mu_M(g_M)} \big(I_M(1)\big)_{(g_M)} m_i, \qquad \big(I_M(1)\big)_{(g_M)} = \bigsqcup_{j=1}^{\nu_M(g)} \mathrm{pr}_M\big((I_P(1))_{(g)}\big) h_j,$$

$$I_{U_P} = \bigsqcup_{s=1}^{\mu_{U_P}(g)} \big(I_{U_P}\big)_{(g)} u_s,$$

it follows that

$$I_P(1) g I_P(1) = \bigsqcup_{i=1}^{\mu_M(g_M)} \bigsqcup_{j=1}^{\nu_M(g)} \bigsqcup_{s=1}^{\mu_{U_P}(g)} I_P(1) g u_s h_j m_i. \tag{3.1.2}$$

In particular, one has $\mu(g) := [I_P(1) : (I_P(1))_{(g)}] = \mu_M(g_M) \cdot \nu_M(g) \cdot \mu_{U_P}(g)$.

Proof. As $I_P(1)$ is compact open in P and $\mathrm{pr}_M : P \twoheadrightarrow M$ is continuous and open, it follows that $\mathrm{pr}_M\big((I_P(1))_{(g)}\big)$ is compact open in M. It is clear that $I_M(1)$ and $(I_M(1))_{(g_M)}$ are compact open in M and that I_{U_P} and $(I_{U_P})_{(g)}$ are compact open in U_P. Therefore, all the indices in question are finite.

Assume now the decompositions of $I_M(1)$, $\big(I_M(1)\big)_{(g_M)}$, and I_{U_P} given as in the statement of the proposition.

It is clear that the right hand side of (3.1.2) is contained in the left hand side. Conversely, let $x \in I_P(1)$. We deduce $x_M \in I_M(1)$ and $x_U \in I_{U_P}$. Hence, there exist $1 \le i \le \mu_M(g_M)$ and $m \in \big(I_M(1)\big)_{(g_M)}$ such that $x_M = mm_i$. Moreover, there exist $h \in \mathrm{pr}_M\big((I_P(1))_{(g)}\big)$ and $1 \le j \le \nu_M(g)$ such that $m = hh_j$. Since $h \in \mathrm{pr}_M\big((I_P(1))_{(g)}\big)$, there exists $u \in I_{U_P}$ such that $hu^{-1} \in \big(I_P(1)\big)_{(g)}$. Thus, $ghu^{-1} = yg$ for some $y \in I_P(1)$. Up to now, we have

$$x = x_M x_U = hh_j m_i x_U = hx_U^{(h_j m_i)^{-1}} h_j m_i = hu^{-1}ux_U^{(h_j m_i)^{-1}} h_j m_i.$$

Since $ux_U^{(h_j m_i)^{-1}} \in I_{U_P}$ we have $ux_U^{(h_j m_i)^{-1}} = vu_s$ for some $v \in \big(I_{U_P}\big)_{(g)}$ and some integer $1 \le s \le \mu_{U_P}(g)$. By definition of v we have $v^{g^{-1}} \in I_{U_P}$. Thus, we compute

$$\begin{aligned} gx &= ghu^{-1}ux_U^{(h_j m_i)^{-1}} h_j m_i = ghu^{-1}vu_s h_j m_i \\ &= ygvu_s h_j m_i = yv^{g^{-1}} \cdot g \cdot u_s h_j m_i \in I_P(1) g u_s h_j m_i. \end{aligned}$$

This computation shows that $I_P(1)gI_P(1)$ is contained on the right hand side of (3.1.2), thus proving equality.

We now prove that the union in the right hand side of (3.1.2) is disjoint. Assume that $gu_s h_j m_i = xgu_t h_a m_b$ for some $x \in I_P(1)$. Rearranging gives

$$x^g = u_s h_j m_i m_b^{-1} h_a^{-1} u_t^{-1} \in I_P(1), \tag{3.1.3}$$

and hence $x^g \in \big(I_P(1)\big)_{(g)}$. Apply pr_M to both sides of (3.1.3) to obtain

$$h_j m_i m_b^{-1} h_a^{-1} = x_M^{g_M} \in \mathrm{pr}_M\big((I_P(1))_{(g)}\big). \tag{3.1.4}$$

In particular, we have

$$m_i m_b^{-1} = h_j^{-1} x_M^{g_M} h_a \in \big(I_M(1)\big)_{(g_M)}.$$

The definition of the coset representatives implies $i = b$. Now, (3.1.4) reads $h_j h_a^{-1} = x_M^{g_M} \in \mathrm{pr}_M\big((I_P(1))_{(g)}\big)$. Again, the definition of the coset representatives yields $j = a$. Going back to (3.1.3), we have shown $u_s u_t^{-1} = x^g \in I_P(1) \cap \big(I_P(1)\big)^g \cap U_P$. Since g normalizes U_P, we see that $I_P(1) \cap \big(I_P(1)\big)^g \cap U_P = I_{U_P} \cap \big(I_{U_P}\big)^g = \big(I_{U_P}\big)_{(g)}$. Put together this gives $u_s u_t^{-1} \in \big(I_{U_P}\big)_{(g)}$, and the definition of the coset representatives shows $s = t$. This concludes the proof of the disjointness assertion. □

Remark 3.6. We make some observations regarding the numbers $\nu_M(g)$ that appeared in Proposition 3.5.

(a) In general, we have $\mathrm{pr}_M\big((I_P(1))_{(g)}\big) \subsetneq \big(I_M(1)\big)_{(g_M)}$, i. e. $\nu_M(g) \neq 1$.

Consider for example $\mathrm{GL}_2(\mathbb{Q}_p)$ and $P = \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ & \mathbb{Q}_p^\times \end{pmatrix}$ with Levi component $M = \begin{pmatrix} \mathbb{Q}_p^\times & \\ & \mathbb{Q}_p^\times \end{pmatrix}$ and unipotent radical $U_P = \begin{pmatrix} 1 & \mathbb{Q}_p \\ & 1 \end{pmatrix}$. Then $I_P(1) = \begin{pmatrix} (1+p\mathbb{Z}_p)^\times & \mathbb{Z}_p \\ & (1+p\mathbb{Z}_p)^\times \end{pmatrix}$ and $I_M(1) = \begin{pmatrix} (1+p\mathbb{Z}_p)^\times & \\ & (1+p\mathbb{Z}_p)^\times \end{pmatrix}$. Let $n \in \mathbb{Z}_{\ge 0}$ be an integer. For $g := \begin{pmatrix} 1 & p^{-n-1} \\ & 1 \end{pmatrix}$ we have $g_M = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and hence $\big(I_M(1)\big)_{(g_M)} = I_M(1)$. Given $x = \begin{pmatrix} 1+pa & b \\ & 1+pc \end{pmatrix} \in I_P(1)$, we compute

$$\begin{aligned} gxg^{-1} &= \begin{pmatrix} 1 & p^{-n-1} \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+pa & b \\ & 1+pc \end{pmatrix} \cdot \begin{pmatrix} 1 & -p^{-n-1} \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+pa & b+p^{-n}(c-a) \\ & 1+pc \end{pmatrix}. \end{aligned}$$

Hence $gxg^{-1} \in I_P(1)$ if and only if $c - a \in p^n\mathbb{Z}_p$. Therefore,

$$\big(I_P(1)\big)_{(g)} = \left\{ \begin{pmatrix} 1+pa & b \\ & 1+pa+p^{n+1}c \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z}_p \right\},$$

and hence $\mathrm{pr}_M\big((I_P(1))_{(g)}\big) = \left\{ \begin{pmatrix} 1+pa & \\ & 1+pa+p^{n+1}c \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z}_p \right\}$. At this point it is already clear that $\nu_M(g) \neq 1$, provided $n \neq 0$.

As an exercise we will compute the index of $\mathrm{pr}_M\big((I_P(1))_{(g)}\big)$ in $I_M(1)$. Consider the reduction modulo p^{n+1} map

$$\varphi\colon \begin{pmatrix} (1+p\mathbb{Z}_p)^\times & \\ & (1+p\mathbb{Z}_p)^\times \end{pmatrix} \longrightarrow \begin{pmatrix} (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^\times & \\ & (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^\times \end{pmatrix}.$$

The kernel of φ equals $\begin{pmatrix} (1+p^{n+1}\mathbb{Z}_p)^\times & \\ & (1+p^{n+1}\mathbb{Z}_p)^\times \end{pmatrix}$ and is contained in the subgroup $\mathrm{pr}_M\big((I_P(1))_{(g)}\big)$. We have

$$\varphi\big(I_M(1)\big) = \left\{ \begin{pmatrix} 1+pa+p^{n+1}\mathbb{Z}_p & \\ & 1+pc+p^{n+1}\mathbb{Z}_p \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z}_p \right\},$$

and hence also $|\varphi(I_M(1))| = p^{2n}$. Moreover, the subgroup

$$\varphi\big(\mathrm{pr}_M\big((I_P(1))_{(g)}\big)\big) = \left\{ \begin{pmatrix} 1+pa+p^{n+1}\mathbb{Z}_p & \\ & 1+pa+p^{n+1}\mathbb{Z}_p \end{pmatrix} \,\middle|\, a \in \mathbb{Z}_p \right\}$$

has order p^n. Put together, we have

$$\nu_M(g) = \big[I_M(1) : \mathrm{pr}_M\big((I_P(1))_{(g)}\big)\big] = \frac{|\varphi(I_M(1))|}{|\varphi(\mathrm{pr}_M((I_P(1))_{(g)}))|} = \frac{p^{2n}}{p^n} = p^n.$$

(b) For each $g \in M$ we have $\nu_M(g) = 1$, because of $(I_M(1))_{(g)} \subseteq (I_P(1))_{(g)}$. Given any $g \in P$, Remark 2.4 implies that μ and μ_{U_P} are constant on $I_P(1)gI_P(1)$ and μ_M is constant on $I_M(1)g_MI_M(1)$. Proposition 3.5 then implies that ν_M is also constant on $I_P(1)gI_P(1)$. In particular, we have $\nu_M\big(I_P(1)MI_P(1)\big) = \{1\}$.

Conversely, given $g \in P$ with $\nu_M(g) = 1$, does it necessarily follow that $g \in I_P(1)MI_P(1)$?

The answer to this question is negative: consider, for instance, $\mathrm{GL}_2(\mathbb{Q}_p)$ and take P, M as in (a). We have $I_P(1)MI_P(1) = I_{U_P}MI_{U_P}$ and therefore also $I_P(1)MI_P(1) \cap U_P = I_{U_P}$. Thus, $g := \begin{pmatrix} 1 & p^{-1} \\ & 1 \end{pmatrix} \notin I_P(1)MI_P(1)$, and the computation in (a) shows $\nu_M(g) = 1$.

(c) Remark 2.4 and Proposition 3.5 together imply that the map

$$P \longrightarrow \mathbb{Q}^{\times}, \quad g \longmapsto \nu_M(g) \cdot \nu_M\big(g^{-1}\big)^{-1}$$

is a locally constant group homomorphism.

Lemma 3.7. *Let $g \in P$ with $\mu_{U_P}(g) = 1$. Then we have $g_M \in M^+$.*

Proof. The condition $\mu_{U_P}(g) = [I_{U_P} : (I_{U_P})_{(g)}] = 1$ is equivalent to $gI_{U_P}g^{-1} \subseteq I_{U_P}$.

Write $g = (A_{ij})_{i,j=1,\dots,r}$ with $A_{ij} \in \mathrm{Mat}_{n_i,n_j}(F)$. We have $A_{ij} = 0$ for $i > j$ and $A_{ii} \in \mathrm{GL}_{n_i}(F)$ for all i. Denote by $g^{-1} = (A^{ij})_{i,j=1,\dots,r}$ the inverse of g. By computing the $(i, i+1)$-matrix of $g \cdot g^{-1}$ we obtain

$$A_{ii}A^{i,i+1} = -A_{i,i+1}A^{i+1,i+1} \quad \text{for all } i = 1, \dots, r-1.$$

Let $u = (B_{ij})_{i,j=1,\dots,r} \in I_{U_P}$ be arbitrary. We have $B_{ii} = E_{n_i}$ and $B_{ij} \in \mathrm{Mat}_{n_i,n_j}(\mathcal{O}_F)$ for all i, j. For each $1 \le i \le r-1$ the $(i, i+1)$-matrix of gug^{-1} equals

$$\begin{aligned}
\sum_{k,l=1}^{r} A_{ik}B_{kl}A^{l,i+1} &= \sum_{i \le k \le l \le i+1} A_{ik}B_{kl}A^{l,i+1} \\
&= A_{ii}E_{n_i}A^{i,i+1} + A_{ii}B_{i,i+1}A^{i+1,i+1} + A_{i,i+1}E_{n_{i+1}}A^{i+1,i+1} \\
&= -A_{i,i+1}A^{i+1,i+1} + A_{ii}B_{i,i+1}A^{i+1,i+1} + A_{i,i+1}A^{i+1,i+1} \\
&= A_{ii}B_{i,i+1}A^{i+1,i+1} = A_{ii}B_{i,i+1}A_{i+1,i+1}^{-1}.
\end{aligned}$$

Hence, $gI_{U_P}g^{-1} \subseteq I_{U_P}$ implies $A_{ii}\,\mathrm{Mat}_{n_i,n_{i+1}}(\mathcal{O}_F)A_{i+1,i+1}^{-1} \subseteq \mathrm{Mat}_{n_i,n_{i+1}}(\mathcal{O}_F)$ for all $1 \le i \le r-1$. But this is equivalent to $d_{n_i}(A_{ii}) \ge d_1(A_{i+1,i+1})$ for all $1 \le i \le r-1$. Now, Lemma 3.3 implies $g_M = \mathrm{diag}(A_{11}, \dots, A_{rr}) \in M^+$. □

Proposition 3.8. *The R-linear map*

$$\begin{aligned}
\Theta_M^P := \Theta_{M,R}^P \colon H_R\big(I_P(1), P\big) &\longrightarrow H_R\big(I_M(1), M\big), \qquad (3.1.5)\\
(g)_{I_P(1)} &\longmapsto \nu_M(g)\mu_{U_P}(g) \cdot (g_M)_{I_M(1)}
\end{aligned}$$

is a homomorphism of R-algebras.

We have $H_R\big(I_M(1), M^+\big) \subseteq \mathrm{Im}(\Theta_M^P)$ with equality whenever $qR = 0$.

Proof. Consider the R-linear map on the universal modules

$$\vartheta \colon X_R\big(I_P(1), P\big) \longrightarrow X_R\big(I_M(1), M\big), \quad \big(I_P(1)g\big) \longmapsto \big(I_M(1)g_M\big).$$

It is well-defined because of $\mathrm{pr}_M(I_P(1)) \subseteq I_M(1)$. ϑ is M-linear with respect to the right action of M on $X_R\big(I_P(1), P\big)$ and $X_R\big(I_M(1), M\big)$ (2.1.1), because of $\mathrm{pr}_M\big|_M = \mathrm{id}_M$. As $I_M(1)$ is contained in $I_P(1)$ restriction gives a well-defined R-linear map

$$\vartheta\big|_{H_R(I_P(1),P)} \colon H_R\big(I_P(1), P\big) \longrightarrow H_R\big(I_M(1), M\big).$$

We show that it coincides with Θ_M^P. Let $g \in P$ be arbitrary. By Proposition 3.5 we may write

$$(g)_{I_P(1)} = \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} \big(I_P(1) g u_s h_j m_i\big)$$

for certain $m_i \in I_M(1)$, $h_j \in \big(I_M(1)\big)_{(g_M)}$ and $u_s \in I_{U_P}$. Notice that $h_j^{g_M^{-1}} \in I_M(1)$, and thus $I_M(1) g_M h_j = I_M(1) g_M$ for all j. We obtain

$$\begin{aligned}
\vartheta\big((g)_{I_P(1)}\big) &= \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} \vartheta\big((I_P(1) g u_s h_j m_i\big) \\
&= \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} \big(I_M(1) g_M h_j m_i\big) \\
&= \sum_{i=1}^{\mu_M(g_M)} \nu_M(g) \mu_{U_P}(g) \cdot \big(I_M(1) g_M m_i\big) \\
&= \nu_M(g) \mu_{U_P}(g) \cdot (g_M)_{I_M(1)} \\
&= \Theta_M^P\big((g)_{I_P(1)}\big).
\end{aligned}$$

For each $m \in M^+$ we have $\nu_M(m) = \mu_{U_P}(m) = 1$, and hence $H_R\big(I_M(1), M^+\big)$ is contained in the image of Θ_M^P. Given $g \in P$ with $\mu_{U_P}(g) = 1$, we have $g_M \in M^+$ by Lemma 3.7. In general, $\nu_M(g)$ and $\mu_{U_P}(g)$ are powers of q. We conclude that the image of Θ_M^P equals the positive subalgebra $H_R\big(I_M(1), M^+\big)$ provided that $qR = 0$.

It remains to show that Θ_M^P is actually a homomorphism of R-algebras. It is clear that Θ_M^P preserves the unit element. Let $g, g' \in P$, and write $(g)_{I_P(1)} = \sum_i \big(I_P(1) g_i\big)$ and $(g')_{I_P(1)} = \sum_j \big(I_P(1) g'_j\big)$. We compute

$$\begin{aligned}
\Theta_M^P\big((g)_{I_P(1)} \cdot (g')_{I_P(1)}\big) &= \sum_{i,j} \vartheta\big((I_P(1) g_i g'_j)\big) = \sum_{i,j} \big(I_M(1) g_{i,M} g'_{j,M}\big) \\
&= \left(\sum_i \big(I_M(1) g_{i,M}\big)\right) \cdot \left(\sum_j \big(I_M(1) g'_{j,M}\big)\right) \\
&= \left(\sum_i \vartheta\big((I_P(1) g_i)\big)\right) \cdot \left(\sum_j \vartheta\big((I_P(1) g'_j)\big)\right) \\
&= \Theta_M^P\big((g)_{I_P(1)}\big) \cdot \Theta_M^P\big((g')_{I_P(1)}\big).
\end{aligned}$$

Therefore, Θ_M^P is a homomorphism of R-algebras. □

Remark 3.9. Proposition 3.8 remains true if we replace $I(1)$ by K. The only difference is that ν_M will take values in $\mathbb{N}$ that are not necessarily q-powers.

By [Vig98, II.5 Prop.] the map

$$\theta^+ \colon H_R\big(I_M(1), M^+\big) \longrightarrow H_R\big(I(1), G\big), \quad (m)_{I_M(1)} \longmapsto (m)_{I(1)}$$

is a homomorphism of R-algebras.[6] Precomposition with Θ_M^P yields the following corollary:

Corollary 3.10. *Assume that $qR = 0$. Then the R-linear map*

$$\begin{aligned}\widetilde{\Theta}_G^P \colon H_R\big(I_P(1), P\big) &\longrightarrow H_R\big(I(1), G\big),\\ (g)_{I_P(1)} &\longmapsto \nu_M(g)\mu_{U_P}(g)\cdot (g_M)_{I(1)}\end{aligned}$$

is a homomorphism of R-algebras.

Remark 3.11. The assumption $qR = 0$ in the above corollary made it possible to give an ad-hoc definition of an R-algebra homomorphism $H_R\big(I_P(1), P\big) \to H_R\big(I(1), G\big)$ in order to define the parabolic induction functor. We will later define a homomorphism $\Xi_G^P \colon H_R\big(I_P(1), P\big) \to H_R\big(I(1), G\big)$ without assuming $qR = 0$, but this one will be quite different from $\widetilde{\Theta}_G^P$ in the case $qR = 0$. Indeed, by construction the image of $\widetilde{\Theta}_G^P$ is just the image of $H_R(I_M(1), M^+)$, whereas the image of Ξ_G^P will be much larger. Nevertheless, the parabolic induction will turn out to be the same.

3.2. The centralizer of a strictly positive element

It is possible to lift the positive subalgebra $H_R\big(I_M(1), M^+\big)$ of $H_R(I_M(1), M)$ to a subalgebra of $H_R\big(I_P(1), P\big)$ by considering so-called *strictly positive elements*.

Definition 3.12. A positive element $a \in M^+$ is called *strictly positive* if it satisfies the following conditions:

- ◇ a is contained in the center $Z(M)$ of M;
- ◇ for all compact open subgroups U_1, U_2 of U_P there exists $k \in \mathbb{Z}_{\geq 0}$ such that $a^k U_1 a^{-k} \subseteq U_2$;
- ◇ for all compact open subgroups U_1, U_2 of $U_{P^{\mathrm{op}}}$ there exists $k \in \mathbb{Z}_{\geq 0}$ such that $a^{-k} U_1 a^k \subseteq U_2$.

Equivalently, as each element of U_P (resp. $U_{P^{\mathrm{op}}}$) is contained in a compact open subgroup of U_P (resp. $U_{P^{\mathrm{op}}}$), the positive element a is strictly positive if $a \in Z(M)$ and it satisfies

$$\bigcap_{k\in\mathbb{Z}_{\geq 0}} a^k I_{U_P} a^{-k} = \{1\} \quad \text{and} \quad \bigcap_{k\in\mathbb{Z}_{\geq 0}} a^{-k} I_{U_{P^{\mathrm{op}}}} a^k = \{1\}.$$

Example 3.13. Given integers $k_1 > k_2 > \cdots > k_r$, the element

$$\begin{pmatrix} \pi_F^{k_1} E_{n_1} & & & \\ & \pi_F^{k_2} E_{n_2} & & \\ & & \ddots & \\ & & & \pi_F^{k_r} E_{n_r} \end{pmatrix} \in Z(M)$$

is strictly positive.

[6] Notice that the result in [Vig98] makes use of the Iwahori decomposition for $I(1)$. In particular, it *cannot* be applied to K instead of $I(1)$.

As in [And77] we fix a strictly positive element $a \in M^+$ and consider the centralizer

$$C(a) \coloneqq \left\{X \in H_R\big(I_P(1), P\big) \,\middle|\, X \cdot (a)_{I_P(1)} = (a)_{I_P(1)} \cdot X\right\} \tag{3.2.1}$$

of $(a)_{I_P(1)}$ inside $H_R\big(I_P(1), P\big)$.

Lemma 3.14. *The subalgebra $C(a)$ of $H_R\big(I_P(1), P\big)$ is a free R-module with basis*

$$B \coloneqq \left\{(g)_{I_P(1)} \in H_R\big(I_P(1), P\big) \,\middle|\, g \in M^+\right\}. \tag{3.2.2}$$

Proof. As a is a central element of M with $aI_{U_P}a^{-1} \subseteq I_{U_P}$, and because of $I_P(1) = I_M(1)I_{U_P}$, we have $I_P(1)aI_P(1) = I_P(1)a$ and hence $(a)_{I_P(1)} = \big(I_P(1)a\big)$.

Let C be the free R-submodule of $H_R\big(I_P(1), P\big)$ generated by B. The claim of the lemma translates to $C = C(a)$.

Given $g \in M^+$, we have $\nu_M(g) = \mu_{U_P}(g) = 1$. Hence, Proposition 3.5 implies $(g)_{I_P(1)} = \sum_{i=1}^{\mu_M(g)} \big(I_P(1)m_i\big)$ for certain $m_i \in M$. We compute

$$(g)_{I_P(1)} \cdot (a)_{I_P(1)} = \sum_{i=1}^{\mu_M(g)} \big(I_P(1)m_i a\big) = \sum_{i=1}^{\mu_M(g)} \big(I_P(1)am_i\big) = (a)_{I_P(1)} \cdot (g)_{I_P(1)}.$$

This shows $(g)_{I_P(1)} \subseteq C(a)$, and consequently $C \subseteq C(a)$.

For the reverse inclusion take $x \in C(a)$ and write $x = \sum_{i=1}^{t} \lambda_i \cdot \big(I_P(1)g_i\big)$ for some $\lambda_i \in R$ and $g_i \in P$. Since a is strictly positive and each element of U_P is contained in a compact open subgroup there exists $k \in \mathbb{Z}_{\geq 0}$ such that

$$a^k g_{i,U}^{g_{i,M}^{-1}} a^{-k} \in I_{U_P} \quad \text{for all } 1 \leq i \leq t.$$

Using the right action of P on the universal module $X_R\big(I_P(1), P\big)$ (2.1.1) we compute

$$\begin{aligned}
x \cdot a^k &= x \cdot (a)_{I_P(1)}^k = (a)_{I_P(1)}^k \cdot x = \sum_{i=1}^{t} \lambda_i \cdot \big(I_P(1)a^k g_{i,M} g_{i,U}\big) \\
&= \sum_{i=1}^{t} \lambda_i \cdot \left(I_P(1)a^k g_{i,U}^{g_{i,M}^{-1}} a^{-k} \cdot a^k g_{i,M}\right) = \sum_{i=1}^{t} \lambda_i \cdot \big(I_P(1)g_{i,M} a^k\big) \\
&= \left(\sum_{i=1}^{t} \lambda_i \cdot \big(I_P(1)g_{i,M}\big)\right) \cdot a^k.
\end{aligned}$$

We conclude $x = \sum_{i=1}^{t} \lambda_i \cdot \big(I_P(1)g_{i,M}\big)$. It remains to show $g_{i,M} \in M^+$ for all $1 \leq i \leq t$.

Let $u \in I_{U_P}$ be arbitrary. Since x is contained in the $I_P(1)$-invariants $H_R\big(I_P(1), P\big)$ of the universal module $X_R\big(I_P(1), P\big)$, we find a permutation $\tau_u \colon \mathfrak{S}_t \longrightarrow \mathfrak{S}_t$ such that $\big(I_P(1)g_{i,M}\big)u^{-1} = \big(I_P(1)g_{\tau_u(i),M}\big)$, i. e. with $I_P(1)g_{i,M}u^{-1} = I_P(1)g_{\tau_u(i),M}$ for all $1 \leq i \leq t$. Hence, there exists $y \in I_P(1)$ such that $yg_{i,M} = g_{\tau_u(i),M}u$. But now we have

$$g_{\tau_u(i),M}u = yg_{i,M} = y_M y_U g_{i,M} = y_M g_{i,M} y_U^{g_{i,M}},$$

and therefore $u = y_U^{g_{i,M}} \in (I_{U_P})_{(g_{i,M})}$ because of $M \cap U_P = \{1\}$. This shows $I_{U_P} \subseteq (I_{U_P})_{(g_{i,M})}$, or, equivalently, $g_{i,M} I_{U_P} g_{i,M}^{-1} \subseteq I_{U_P}$. By Lemma 3.3 this suffices to conclude that $g_{i,M}$ is positive for each $1 \leq i \leq t$. Thus, x is contained in C. This proves the lemma. □

Corollary 3.15. *The homomorphism Θ_M^P (3.1.5) restricts to an isomorphism of R-algebras*

$$\Theta_M^P\big|_{C(a)} \colon C(a) \longrightarrow H_R\big(I_M(1), M^+\big).$$

Proof. We have $\nu_M(g) = \mu_{U_P}(g) = 1$ for $g \in M^+$. Therefore, Θ_M^P identifies the basis B of $C(a)$ (3.2.2) with the canonical basis of $H_R\big(I_M(1), M^+\big)$. □

4. Parabolic induction for a connected reductive group

We keep the notations of section 2.2. Recall that F is a local field with finite residue field κ_F of cardinality q and characteristic $p > 0$. We denote by R a commutative ring with 1.

Consider a connected reductive group $\mathbf{G}$ over F and a maximal F-split torus $\mathbf{T}$. Let $\mathbf{N} := \mathbf{N}_{\mathbf{G}}(\mathbf{T})$ (resp. $\mathbf{Z} := \mathbf{Z}_{\mathbf{G}}(\mathbf{T})$) be the normalizer (resp. centralizer) of $\mathbf{T}$ in $\mathbf{G}$. The finite Weyl group $W_0 = N/Z$ is the Weyl group of the relative root system $\Phi := \Phi(\mathbf{G}, \mathbf{T})$. Denote by $\mathbf{C}$ the connected center of $\mathbf{G}$. We associated with G a finite-dimensional $\mathbb{R}$-vector space $V = \big(X_*(T)/X_*(C)\big) \otimes_{\mathbb{Z}} \mathbb{R}$ (1.1.2) and an apartment $\mathscr{A}$ of G (1.4.1), which is an affine space under V, together with a system of hyperplanes $\mathfrak{H}$ (1.4.3). We fix a fundamental alcove $\mathfrak{C}$ in $\mathscr{A}$ and a special point $\varphi_0 \in \overline{\mathfrak{C}}$. We fix the unique basis Δ of Φ such that the associated set of positive roots Φ^+ coincides with $\{\alpha \in \Phi \mid \langle \alpha, x - \varphi_0 \rangle > 0\}$ for some arbitrarily chosen $x \in \mathfrak{C}$. Let $K \subseteq G$ be the parahoric subgroup of G attached to φ_0, and let $I \subseteq G$ be the Iwahori subgroup attached to $\mathfrak{C}$ with pro-p radical $I(1)$. The Iwahori-Weyl group $W := N/Z_0$ (resp. the pro-p Iwahori-Weyl group $W(1) := N/Z_0(1)$) parametrizes the set of double cosets $I\backslash G/I$ (resp. $I(1)\backslash G/I(1)$) and acts on $\mathscr{A}$ and $\mathfrak{H}$ via the natural action $\nu\colon N \longrightarrow \operatorname{Aut}\mathscr{A}$ (Proposition 1.27). The kernel Λ of the natural projection map $W \twoheadrightarrow W_0$ is a finitely generated abelian group. We denote by Σ the unique reduced root system corresponding to the affine Weyl group W^{aff} (Proposition 1.32). There is a surjective map $\Phi \twoheadrightarrow \Sigma$ inducing a bijection $\Phi_{\mathrm{red}} \cong \Sigma$. Let Π be the basis of Σ corresponding to Δ under this bijection.

We denote by $\mathcal{H}_R(G) := H_R\big(I(1), G\big)$ the pro-p Iwahori-Hecke algebra of G. We write $T_w := T_g := \big(I(1)gI(1)\big)$ for $w \in W(1)$ representing the double coset $I(1)gI(1)$.

As in section 2.3 we choose a (standard) Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ associated with a subset $J \subseteq \Delta$. Then J is a basis of $\Phi_M := \Phi(\mathbf{M}, \mathbf{T})$ and $\mathbf{M}$ is itself a connected reductive group over F, hence all the results in section 1 apply to M. We denote by $W_{0,M}$, W_M, and $W_M(1)$ the finite, Iwahori-Weyl, and pro-p Iwahori-Weyl group of M. Write $\Psi := \Phi^+ \smallsetminus \Phi_M$ and denote by Ψ_{red} the set of reduced roots in Ψ. Let $\mathbf{P}$ (resp. $\mathbf{P}^{\mathrm{op}}$) be the (standard) parabolic subgroup containing $\mathbf{M}$ with unipotent radical $\mathbf{U}_P := \prod_{\alpha\in\Psi_{\mathrm{red}}} \mathbf{U}_\alpha$ (resp. $\mathbf{U}_{P^{\mathrm{op}}} := \prod_{\alpha\in-\Psi_{\mathrm{red}}} \mathbf{U}_\alpha$). We have $P = M \ltimes U_P$ and $P^{\mathrm{op}} = M \ltimes U_{P^{\mathrm{op}}}$, and $I_M = I \cap M$ (resp. $I_M(1) = I(1) \cap M$) is the Iwahori subgroup (resp. pro-p Iwahori subgroup) of M. We write $I_{U_{P^{\mathrm{op}}}} := I \cap U_{P^{\mathrm{op}}} = I(1) \cap U_{P^{\mathrm{op}}}$ and $I_{U_P} := I \cap U_P = I(1) \cap U_P$. We have a decomposition $I(1) = I_{U_{P^{\mathrm{op}}}} I_M(1) I_{U_P}$, with respect to any order of the three factors, by Lemma 1.58.

We denote by $\mathcal{H}_R(M) = H_R\big(I_M(1), M\big)$ the pro-p Iwahori-Hecke algebra of M. We write $T^M_w := T^M_g := \big(I_M(1)gI_M(1)\big)$ for $w \in W_M(1)$ representing the double coset $I_M(1)gI_M(1)$.

Let M^+ be the submonoid of M consisting of positive elements (1.8.2). Recall that an element $m \in M$ is positive if it satisfies $mI_{U_P}m^{-1} \subseteq I_{U_P}$ and $I_{U_{P^{\mathrm{op}}}} \subseteq mI_{U_{P^{\mathrm{op}}}}m^{-1}$. If we let $W_{M^+} := \big\{w \in W_M \,\big|\, w(\Sigma^+ \smallsetminus \Sigma_M) \subseteq \Sigma^{\mathrm{aff},+}\big\}$, then we have the Bruhat decompositions $W_{M^+} \cong I_M\backslash M^+/I_M$ and $W_{M^+}(1) \cong I_M(1)\backslash M^+/I_M(1)$, and a semidirect product decomposition $W_{M^+} \cong \Lambda_{M^+} \rtimes W_{0,M}$, where $\Lambda_{M^+} := \{\lambda \in \Lambda \mid \langle \alpha, \nu(\lambda) \rangle \leq 0 \text{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_M\}$, see Proposition 1.62.

We denote by $\mathcal{H}_R(M^+) = H_R\big(I_M(1), M^+\big)$ the positive subalgebra of $\mathcal{H}_R(M)$.

Definition 4.1. The R-algebra $H_R\big(I_P(1), P\big)$, where $I_P(1) := I(1) \cap P$, is called the *parabolic Hecke algebra* associated with the parabolic subgroup P of G.

The goal of this section is to extend the results in section 3 to the connected reductive group $\mathbf{G}$ and the commutative ring R with 1 (i. e. without requiring $qR = 0$). More concretely, we will define homomorphisms of R-algebras

$$\begin{tikzcd} & H_R\big(I_P(1), P\big) \arrow[dl, "\Xi_G^P"'] \arrow[dr, "\Theta_M^P"] & \\ \mathcal{H}_R(G) & & \mathcal{H}_R(M). \end{tikzcd}$$

Using these morphisms we consider the functor $- \otimes_{H_R(I_P(1),P)} \mathcal{H}_R(G)$ from the category of right modules over $\mathcal{H}_R(M)$ to the category of right modules over $\mathcal{H}_R(G)$ and show that it coincides with the parabolic induction studied in [Vig15] and [OV18].

4.1. Construction of Θ_M^P

Let $\Gamma \subseteq P$ be a compact open subgroup with $\Gamma = \Gamma_M \Gamma_{U_P}$, where $\Gamma_M := \Gamma \cap M$ and $\Gamma_{U_P} := \Gamma \cap U_P = I_{U_P}$. For example, we might take $\Gamma = I_P(1)$ or $\Gamma = K \cap P$. Since Γ_M normalizes Γ_{U_P}, we deduce $\Gamma_M \subseteq M^+$ from Corollary 1.63. We use this setting throughout sections 4.1 and 4.2.

The existence of the map $\Theta_M^P : H_R\big(\Gamma, P\big) \to H_R(\Gamma_M, M)$ follows from a simple translation of the arguments in section 3.1.

We recall Notation 3.4: given a group H, a subgroup H', and elements $g, h \in H$, we write

$$h^g := g^{-1}hg, \quad H'^g := \big\{h^g \,\big|\, h \in H'\big\}, \quad \text{and} \quad H'_{(g)} := H'^g \cap H'.$$

Also, given $g \in P$ and $m \in M$ we let $\mu(g) := [\Gamma : \Gamma_{(g)}]$, $\mu_{U_P}(g) := [I_{U_P} : (I_{U_P})_{(g)}]$ and $\mu_M(m) := [\Gamma_M : (\Gamma_M)_{(m)}]$. These indices are finite, because Γ is compact open in P. Notice that the projection map $\mathrm{pr}_M : P \twoheadrightarrow M$ with kernel U_P is continuous and open. Given $g \in P$, we write $g_M := \mathrm{pr}_M(g) \in M$ and $g_U := g_M^{-1} g \in U_P$, so that we have a unique decomposition

$$g = g_M g_U \qquad \text{with } g_M \in M \text{ and } g_U \in U_P.$$

Proposition 4.2. *Let $g \in P$ and write*

$$\Gamma_M = \bigsqcup_{i=1}^{\mu_M(g_M)} (\Gamma_M)_{(g_M)} m_i, \quad (\Gamma_M)_{(g_M)} = \bigsqcup_{j=1}^{\nu_M(g)} \mathrm{pr}_M\big(\Gamma_{(g)}\big) h_j, \quad \Gamma_{U_P} = \bigsqcup_{s=1}^{\mu_{U_P}(g)} (\Gamma_{U_P})_{(g)} u_s.$$

Then we have

$$\Gamma g \Gamma = \bigsqcup_{i=1}^{\mu_M(g_M)} \bigsqcup_{j=1}^{\nu_M(g)} \bigsqcup_{s=1}^{\mu_{U_P}(g)} \Gamma g u_s h_j m_i,$$

and in particular $\mu(g) = \mu_M(g_M) \cdot \nu_M(g) \cdot \mu_{U_P}(g)$.

Proof. See Proposition 3.5. The proof applies in this generality. □

Proposition 4.3. *The map*

$$\Theta^P_M : H_R(\Gamma, P) \longrightarrow H_R(\Gamma_M, M), \quad (g)_\Gamma \longmapsto \nu_M(g)\mu_{U_P}(g) \cdot (g_M)_{\Gamma_M}$$

is a homomorphism of R-algebras. The image of Θ^P_M contains $H_R(\Gamma_M, M^+)$.

Proof. See the first statement of Proposition 3.8. The proof applies in this generality. □

Remark 4.4. In Proposition 4.3 we have not (yet) claimed that the image of Θ^P_M coincides with $H_R(\Gamma_M, M^+)$ provided $qR = 0$. The reason is that we have not proved an analogue of Lemma 3.7, which states that for each $g \in P$ with $\mu_{U_P}(g) = 1$ we have $g_M \in M^+$, and whose proof relies on the fact that the underlying group is $\mathrm{GL}_n(F)$. It is possible to give an ad hoc proof of this fact using the methods in section 1.2 (on root group data) and section 1.8 (on positive elements, especially Proposition 1.62).

We omit this for the following reason: the construction of the homomorphism $\Xi^P_G : H_R(I_P(1), P) \to \mathcal{H}_R(G)$ in section 4.3 requires some preliminary work, and the next section is devoted to proving more generally that $\mu_{U_P}(g) \geq \mu_{U_P}(g_M)$ for all $g \in P$. This fact, together with Corollary 1.63, then implies that $\operatorname{Im} \Theta^P_M = H_R(\Gamma_M, M^+)$ whenever $qR = 0$.

4.2. The inequality $\mu_{U_P}(g) \geq \mu_{U_P}(g_M)$

Retain the notations from section 4.1.

The aim of this section is to prove an inequality of indices. We try to adapt the method used for computing $\nu_M(g)$ in Remark 3.6, (a). However, the following fact drastically complicates the proof: As $\mathbf{G}$ is not assumed to be split over F, the root system Φ need not be reduced. This means that there may be non-trivial root groups $U_{2\alpha}$ for $\alpha, 2\alpha \in \Phi$. To deal with these we define a special kind of filtered groups.

Definition 4.5. Let $\mathbf{FGrp}_1$ be the category, whose

- ◇ objects, called *filtered groups*, are pairs (Γ_0, Γ_1) of groups such that Γ_0 is a normal subgroup of Γ_1;
- ◇ morphisms $(\Gamma_0, \Gamma_1) \to (\Gamma'_0, \Gamma'_1)$ between the objects (Γ_0, Γ_1) and (Γ'_0, Γ'_1) are the group homomorphisms $f : \Gamma_1 \to \Gamma'_1$ with $f(\Gamma_0) \subseteq \Gamma'_0$.

For objects (Γ_0, Γ_1) and (Γ'_0, Γ'_1) of $\mathbf{FGrp}_1$, we write $(\Gamma_0, \Gamma_1) \subseteq (\Gamma'_0, \Gamma'_1)$ if $\Gamma_1 \subseteq \Gamma'_1$ and also $\Gamma_0 = \Gamma_1 \cap \Gamma'_0$.

We have a functor $\mathrm{gr}\colon \mathbf{FGrp}_1 \to \mathbf{Grp}$ into the category of groups given by

$$\mathrm{gr}(\Gamma_0, \Gamma_1) := \mathrm{gr}_0(\Gamma_0, \Gamma_1) \times \mathrm{gr}_1(\Gamma_0, \Gamma_1),$$

where $\mathrm{gr}_0(\Gamma_0, \Gamma_1) := \Gamma_0$ and $\mathrm{gr}_1(\Gamma_0, \Gamma_1) := \Gamma_1/\Gamma_0$.

Lemma 4.6. *Let (Γ_0, Γ_1) and (Γ'_0, Γ'_1) be filtered groups with $(\Gamma_0, \Gamma_1) \subseteq (\Gamma'_0, \Gamma'_1)$.*

(a) *We have* $\mathrm{gr}(\Gamma_0, \Gamma_1) \subseteq \mathrm{gr}(\Gamma_0', \Gamma_1')$.
(b) *Assume that* $[\Gamma_1' : \Gamma_1] < \infty$. *Then we have* $[\Gamma_1' : \Gamma_1] = [\mathrm{gr}(\Gamma_0', \Gamma_1') : \mathrm{gr}(\Gamma_0, \Gamma_1)]$.

Proof. (a) By definition, $(\Gamma_0, \Gamma_1) \subseteq (\Gamma_0', \Gamma_1')$ means $\Gamma_1 \subseteq \Gamma_1'$ and $\Gamma_0 = \Gamma_1 \cap \Gamma_0'$. In particular, we have an inclusion $\Gamma_1/\Gamma_0 \cong (\Gamma_1\Gamma_0')/\Gamma_0' \subseteq \Gamma_1'/\Gamma_0'$. Together with $\Gamma_0 \subseteq \Gamma_0'$ this shows the assertion.

(b) Consider the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_0 \arrow[r] \arrow[d] & \Gamma_1 \arrow[r] \arrow[d] & \Gamma_1/\Gamma_0 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Gamma_0' \arrow[r] \arrow[d] & \Gamma_1' \arrow[r] \arrow[d] & \Gamma_1'/\Gamma_0' \arrow[r] \arrow[d] & 1 \\
& \Gamma_0'/\Gamma_0 \arrow[r, hook, "\alpha"] & \Gamma_1'/\Gamma_1 \arrow[r, two heads, "\beta"] & (\Gamma_1'/\Gamma_0')/(\Gamma_1/\Gamma_0) &
\end{tikzcd}$$

whose first two rows are exact and where the last row consists of sets and set maps only. Notice that $[\mathrm{gr}(\Gamma_0', \Gamma_1') : \mathrm{gr}(\Gamma_0, \Gamma_1)] = [\Gamma_0' : \Gamma_0] \cdot [\Gamma_1'/\Gamma_0' : \Gamma_1/\Gamma_0]$. We write $s := [\Gamma_0' : \Gamma_0]$ and $t := [\Gamma_1'/\Gamma_0' : \Gamma_1/\Gamma_0]$. We need to show $[\Gamma_1' : \Gamma_1] = st$.

It is clear that β is surjective, and hence we have $t \leq [\Gamma_1' : \Gamma_1] < \infty$. Using $\Gamma_0 = \Gamma_0' \cap \Gamma_1$, it is easy to see that α is injective: indeed, given $g, h \in \Gamma_0'$ with

$$g\Gamma_1 \overset{\text{def}}{=} \alpha(g\Gamma_0) = \alpha(h\Gamma_0) \overset{\text{def}}{=} h\Gamma_1,$$

we find $\gamma \in \Gamma_1$ with $g = h\gamma$. But then $\gamma = h^{-1}g \in \Gamma_1 \cap \Gamma_0' = \Gamma_0$, i. e. $g\Gamma_0 = h\Gamma_0$. Hence also $s \leq [\Gamma_1' : \Gamma_1] < \infty$. To prove the claim, we write

$$\Gamma_0' = \bigsqcup_{i=1}^{s} \sigma_i\Gamma_0 \qquad \text{and} \qquad \Gamma_1'/\Gamma_0' = \bigsqcup_{j=1}^{t} (\gamma_j\Gamma_0').(\Gamma_1/\Gamma_0)$$

for certain $\sigma_1, \ldots, \sigma_s \in \Gamma_0'$ and $\gamma_1, \ldots, \gamma_t \in \Gamma_1'$. Recall that we identify Γ_1/Γ_0 with $(\Gamma_1\Gamma_0')/\Gamma_0'$ as a subset of Γ_1'/Γ_0'. We claim that we have a disjoint union

$$\Gamma_1' = \bigsqcup_{i=1}^{s} \bigsqcup_{j=1}^{t} \gamma_j\sigma_i\Gamma_1. \tag{4.2.1}$$

It is clear that the right hand side is contained in Γ_1'. Conversely, let $g' \in \Gamma_1'$ be arbitrary. Let $j \in \{1, \ldots, t\}$ such that $g'\Gamma_0' \in (\gamma_j\Gamma_0').(\Gamma_1/\Gamma_0)$. Then there exists $\gamma \in \Gamma_1$ such that $g'\Gamma_0' = (\gamma_j\Gamma_0')\cdot(\gamma\Gamma_0') = \gamma_j\gamma\Gamma_0'$. In other words, we have $g' = \gamma_j\gamma h$ for some $h \in \Gamma_0'$. Since Γ_0' is normal in Γ_1', it follows that $\gamma h\gamma^{-1} \in \Gamma_0'$. Therefore, we find $i \in \{1, \ldots, s\}$ and $\sigma \in \Gamma_0$ with $\gamma h\gamma^{-1} = \sigma_i\sigma$. Taken together, we obtain $g' = \gamma_j\sigma_i\sigma\gamma$ with $\sigma\gamma \in \Gamma_1$, i. e. $g'\Gamma_1 = \gamma_j\sigma_i\Gamma_1$. This establishes the equality in (4.2.1).

It remains to prove disjointness. Suppose we have $\gamma_j\sigma_i\Gamma_1 = \gamma_b\sigma_a\Gamma_1$ for some $a, i \in \{1, \ldots, s\}$ and $b, j \in \{1, \ldots, t\}$. Let $\gamma \in \Gamma_1$ with $\gamma_j\sigma_i = \gamma_b\sigma_a\gamma$. In particular, we have the equality $\gamma_j\Gamma_0' = (\gamma_b\Gamma_0')\cdot(\gamma\Gamma_0')$ and therefore $(\gamma_j\Gamma_0') \in (\gamma_b\Gamma_0').(\Gamma_1/\Gamma_0)$. We conclude that $j = b$. Canceling γ_j in the hypothesis gives $\sigma_i\Gamma_1 = \sigma_a\Gamma_1$. This means $\sigma_a^{-1}\sigma_i \in \Gamma_1 \cap \Gamma_0' = \Gamma_0$, i. e. $\sigma_i\Gamma_0 = \sigma_a\Gamma_0$. We conclude that $i = a$. This establishes the disjointness in (4.2.1).

Finally, (4.2.1) shows $[\Gamma_1' : \Gamma_1] = st = [\Gamma_0' : \Gamma_0]\cdot[\Gamma_1'/\Gamma_0' : \Gamma_1/\Gamma_0] = [\mathrm{gr}(\Gamma_0', \Gamma_1') : \mathrm{gr}(\Gamma_0, \Gamma_1)]$, and this completes the proof. □

Proposition 4.7. *For each $g \in P$ the inequality $\mu_{U_P}(g) \geq \mu_{U_P}(g_M)$ holds.*

Proof. Let $g \in P$. Recall that the multiplication map induces a homeomorphism $\prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,0)} \cong \Gamma_{U_P}$ (see Remark 1.59) and that $\mu_{U_P}(g) = [\Gamma_{U_P} : \Gamma_{U_P} \cap g^{-1}\Gamma_{U_P} g]$. It was remarked in Definition 1.60 that $K_M = K \cap M$ normalizes Γ_{U_P}. Hence, Remark 2.4, (a) shows that the map $\mu_{U_P} : P \to q^{\mathbb{Z}_{\geq 0}}$ is constant on $K_M g K_M$. By Remark 1.45 the group K_M contains representatives of $W_{0,M}$. We conclude that $K_M g_M K_M \cap Z \neq \emptyset$, and hence we may assume $g_M \in Z$.

For each $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$ we have $g_M^{-1} U_{(\alpha,0)} g_M = U_{(\alpha,\langle\alpha,\nu(g_M)\rangle)}$ (1.5.4) and therefore

$$(U_{(\alpha,0)})_{(g_M)} = \begin{cases} U_{(\alpha,0)}, & \text{if } \langle\alpha,\nu(g_M)\rangle \leq 0 \\ U_{(\alpha,0)}^{g_M}, & \text{else.} \end{cases} \tag{4.2.2}$$

Recall the following partial order on $\Psi = \Phi^+ \smallsetminus \Phi_M$ (Definition 1.7): given $\alpha, \beta \in \Psi$, we write $\alpha \leq \beta$ if there exist $\gamma_1, \ldots, \gamma_n \in \Psi$ and $r \in \mathbb{N}$, $s_1, \ldots, s_n \in \mathbb{Z}_{\geq 0}$ with $\beta = r\alpha + \sum_{i=1}^n s_i\gamma_i$.

We choose an ordering $o : \Psi_{\mathrm{red}} \to \{1, 2, \ldots, |\Psi_{\mathrm{red}}|\}$ of the factors of $\prod_{\alpha\in\Psi_{\mathrm{red}}} U_\alpha$ in such a way that $\beta < \alpha$ implies $o(\beta) < o(\alpha)$.

We consider the automorphism $f : U_P \to U_P$, $x \mapsto g_U^{-1} x g_U$.

Claim 1. *The map f satisfies $f(x_\alpha)x_\alpha^{-1} \in \langle U_\beta \mid \beta \in \Psi_{\mathrm{red}}, \beta > \alpha\rangle$ for all $x_\alpha \in U_\alpha$ and $\alpha \in \Psi_{\mathrm{red}}$.*

Proof of the claim. Write $g_U = u_{\alpha_1}\cdots u_{\alpha_r}$ for certain $u_{\alpha_i} \in U_{\alpha_i}$, $i = 1, \ldots, r$, and $\alpha_1, \ldots, \alpha_r \in \Psi_{\mathrm{red}}$. We do an induction over r. For $r = 1$ the claim follows from axiom (DR2) of a root group datum. Now, let $r \in \mathbb{N}_{>1}$ and assume

$$y := (u_{\alpha_1}\cdots u_{\alpha_{r-1}})^{-1} x_\alpha (u_{\alpha_1}\cdots u_{\alpha_{r-1}}) x_\alpha^{-1} \in \langle U_\beta \mid \beta \in \Psi_{\mathrm{red}}, \beta > \alpha\rangle.$$

Again from axiom (DR2) it follows that $y^{u_{\alpha_r}} = u_{\alpha_r}^{-1} y u_{\alpha_r} \in \langle U_\beta \mid \beta \in \Psi_{\mathrm{red}}, \beta > \alpha\rangle$. We compute

$$\begin{aligned} g_U^{-1} x_\alpha g_U x_\alpha^{-1} &= u_{\alpha_r}^{-1}(u_{\alpha_1}\cdots u_{\alpha_{r-1}})^{-1} x_\alpha (u_{\alpha_1}\cdots u_{\alpha_{r-1}}) u_{\alpha_r} x_\alpha^{-1} \\ &= u_{\alpha_r}^{-1} y x_\alpha u_{\alpha_r} x_\alpha^{-1} = y^{u_{\alpha_r}} \cdot u_{\alpha_r}^{-1} x_\alpha u_{\alpha_r} x_\alpha^{-1}, \end{aligned}$$

and this is contained in $\langle U_\beta \mid \beta \in \Psi_{\mathrm{red}}, \beta > \alpha\rangle$. This proves the claim. □

By Claim 1 we may now apply Proposition 1.9: for each $(x_\alpha)_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}} U_\alpha$ we find $z_\alpha(x_\beta)_{\beta<\alpha} \in U_\alpha$ (depending only on $(x_\beta)_{\beta<\alpha}$) and a group homomorphism

$\tilde{z}_\alpha\colon U_\alpha \to U_{2\alpha}$ factoring over $U_\alpha/U_{2\alpha}$ (not depending on $(x_\beta)_{\beta<\alpha}$ by our choice of o; see Remark 1.10) such that

$$f\left(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\right) = \prod_{\alpha\in\Psi_{\mathrm{red}}} \big(z_\alpha(x_\beta)_{\beta<\alpha}\cdot \tilde{z}_\alpha(x_\alpha)\cdot x_\alpha\big).$$

We identify Ψ_{red} with $\Sigma^+ \smallsetminus \Sigma_M$. For $\alpha \in \Psi_{\mathrm{red}}$ we consider the homomorphism

$$f_\alpha\colon U^{g_M}_{(\alpha,0)} \longrightarrow U_\alpha, \quad x^{g_M} \longmapsto \tilde{z}_\alpha(x^{g_M})\cdot x^{g_M}.$$

Claim 2. *The subgroup* $f_\alpha\big(U^{g_M}_{(\alpha,0)}\big)$ *is open in* U_α.

Proof of the claim. Let $\Psi' := \{\beta \in \Psi_{\mathrm{red}} \mid \beta \not< \alpha\}$ and let $Z_\alpha \subseteq U_P$ be the subgroup generated by $\bigcup_{\beta\in\Psi'} U_\beta$. As in the proof of Claim 1.2 in Lemma 1.8 the subgroup Z_α is normal in U_P and the multiplication map induces a homeomorphism $\prod_{\beta\in\Psi'} U_\beta \cong Z_\alpha$. Let $\mathrm{pr}_\alpha\colon Z_\alpha \to U_\alpha$ be the projection map. By normality the inner automorphism f induces an automorphism $f' := f\big|_{Z_\alpha}$ on Z_α. Notice that both f' and pr_α are open maps. Therefore, the subset

$$f_\alpha\big(U^{g_M}_{(\alpha,0)}\big) = (\mathrm{pr}_\alpha \circ f')\left(U^{g_M}_{(\alpha,0)} \times \prod_{\beta\in\Psi'\smallsetminus\{\alpha\}} U_{(\beta,0)}\right) \subseteq U_\alpha$$

is open. □

Claim 3. *We have* $[U_{(\alpha,0)} : f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}] \geq [U_{(\alpha,0)} : (U_{(\alpha,0)})_{(g_M)}]$.

Proof of the claim. Notice that $(U_{2\alpha}, U_\alpha)$ is a filtered group in the sense of Definition 4.5. For each subgroup $H \subseteq U_\alpha$ we have $(U_{2\alpha}\cap H, H) \subseteq (U_{2\alpha}, U_\alpha)$; to simplify notation we write H instead of $(U_{2\alpha}\cap H, H)$.

Step 1: We show $\mathrm{gr}_0\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big) = \mathrm{gr}_0\big((U_{(\alpha,0)})_{(g_M)}\big)$. Equivalently, we need to show

$$f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)} \cap U_{2\alpha} = (U_{(\alpha,0)})_{(g_M)} \cap U_{2\alpha}. \tag{4.2.3}$$

Let $x \in U_{(\alpha,0)}$ such that $f_\alpha(x^{g_M}) = \tilde{z}_\alpha(x^{g_M})\cdot x^{g_M} \in U_{(\alpha,0)} \cap U_{2\alpha}$. Because of $\tilde{z}_\alpha(x^{g_M}) \in U_{2\alpha}$ we have $x^{g_M} \in U_{2\alpha}$, and hence $f_\alpha(x^{g_M}) = x^{g_M}$ since $\tilde{z}_\alpha$ vanishes on $U_{2\alpha}$. Conversely, given $x \in U_{(\alpha,0)}$ with $x^{g_M} \in U_{(\alpha,0)}\cap U_{2\alpha}$, we have $x^{g_M} = f_\alpha(x^{g_M}) \in f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)} \cap U_{2\alpha}$. This proves the equality in (4.2.3).

Step 2: We show $\mathrm{gr}_1\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big) \subseteq \mathrm{gr}_1\big((U_{(\alpha,0)})_{(g_M)}\big)$. We first claim

$$\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big)\cdot U_{2\alpha} \subseteq U^{g_M}_{(\alpha,0)}U_{2\alpha} \cap U_{(\alpha,0)}U_{2\alpha} = (U_{(\alpha,0)})_{(g_M)}\cdot U_{2\alpha}. \tag{4.2.4}$$

The equality is a consequence of (4.2.2). By definition of f_α we have $f_\alpha(U^{g_M}_{(\alpha,0)})U_{2\alpha} =$

$U^{g_M}_{(\alpha,0)}U_{2\alpha}$. Hence, also the inclusion in (4.2.4) holds. We now compute

$$\begin{aligned}
\mathrm{gr}_1\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big) &= \frac{f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}}{f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)} \cap U_{2\alpha}} \\
&= \frac{\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big) \cdot U_{2\alpha}}{U_{2\alpha}} \\
&\subseteq \frac{(U_{(\alpha,0)})_{(g_M)} \cdot U_{2\alpha}}{U_{2\alpha}} \\
&= \frac{(U_{(\alpha,0)})_{(g_M)}}{(U_{(\alpha,0)})_{(g_M)} \cap U_{2\alpha}} \\
&= \mathrm{gr}_1\big((U_{(\alpha,0)})_{(g_M)}\big).
\end{aligned}$$

Step 3: Proof of the claim. Steps 1 and 2 together imply $\mathrm{gr}\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big) \subseteq \mathrm{gr}\big((U_{(\alpha,0)})_{(g_M)}\big)$. Notice that the indices of $(U_{(\alpha,0)})_{(g_M)}$ and $f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}$ in $U_{(\alpha,0)}$ are finite (see Claim 2). By Lemma 4.6, (b) we have

$$\begin{aligned}
\big[U_{(\alpha,0)} : f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big] &= \big[\mathrm{gr}(U_{(\alpha,0)}) : \mathrm{gr}\big(f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\big)\big] \\
&\geq \big[\mathrm{gr}(U_{(\alpha,0)}) : \mathrm{gr}\big((U_{(\alpha,0)})_{(g_M)}\big)\big] \\
&= \big[U_{(\alpha,0)} : (U_{(\alpha,0)})_{(g_M)}\big],
\end{aligned}$$

which proves the claim. □

For all $\alpha \in \Psi_{\mathrm{red}}$ and all $(x_\beta)_{\beta<\alpha} \in \prod_{\beta<\alpha} U_\beta$ we consider the subset

$$\Gamma(x_\beta)_{\beta<\alpha} := U_{(\alpha,0)} \cap \left\{ z_\alpha(x_\beta^{g_M})_{\beta<\alpha} \cdot \tilde{z}_\alpha(x_\alpha^{g_M}) \cdot x_\alpha^{g_M} \,\middle|\, x_\alpha \in U_{(\alpha,0)} \right\} \subseteq U_{(\alpha,0)}.$$

Claim 4. *The set* $\Gamma(x_\beta)_{\beta<\alpha}$ *is either empty or a left coset of* $f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}$.

Proof of the claim. It is clear that $\Gamma(x_\beta)_{\beta<\alpha}$ is stable under right multiplication by elements of $f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}$. Now, take

$$\gamma_i := z_\alpha(x_\beta^{g_M})_{\beta<\alpha} \cdot \tilde{z}_\alpha(x_i^{g_M}) \cdot x_i^{g_M} \in \Gamma(x_\beta)_{\beta<\alpha} \quad \text{with } x_i \in U_{(\alpha,0)},\ i = 1,2.$$

We have $\gamma_2^{-1}\gamma_1 = f_\alpha\big((x_2^{-1}x_1)^{g_M}\big) \in f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}$. This proves the claim. □

The reason why we consider $\Gamma(x_\beta)_{\beta<\alpha}$ is that we can write

$$(\Gamma_{U_P})_{(g)} = \left\{ \prod_{\alpha\in\Psi_{\mathrm{red}}} x'_\alpha = f\left(\prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha\right) \in \prod_{\alpha\in\Psi_{\mathrm{red}}} U_{(\alpha,0)} \,\middle|\, \begin{array}{l} \prod_{\alpha\in\Psi_{\mathrm{red}}} x_\alpha \in \prod_{\alpha\in\Psi_{\mathrm{red}}} U_{(\alpha,0)} \text{ and} \\ x'_\alpha \in \Gamma(x_\beta)_{\beta<\alpha} \text{ for all } \alpha \in \Psi_{\mathrm{red}} \end{array} \right\}.$$

Choose $r \in \mathbb{N}$ such that $U_{(\alpha,r)}$ is contained in $f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)} \cap U^{g_M}_{(\alpha,0)}$ for all $\alpha \in \Psi_{\mathrm{red}}$ and such that $H := \prod_{\alpha\in\Psi_{\mathrm{red}}} U_{(\alpha,r)} \subseteq \Gamma_{U_P}$ is contained in $(\Gamma_{U_P})_{(g)} \cap (\Gamma_{U_P})_{(g_M)}$. Notice that H is a normal subgroup of Γ_{U_P} by axiom (V3) of a valuation of a root group datum. By Claim 3 we have

$$\left|f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}/U_{(\alpha,r)}\right| = \frac{\left|U_{(\alpha,0)}/U_{(\alpha,r)}\right|}{\left[U_{(\alpha,0)} : f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}\right]}$$
$$\leq \frac{\left|U_{(\alpha,0)}/U_{(\alpha,r)}\right|}{\left[U_{(\alpha,0)} : (U_{(\alpha,0)})_{(g_M)}\right]} = \left|(U_{(\alpha,0)})_{(g_M)}/U_{(\alpha,r)}\right|.$$

Using the above presentation of $(\Gamma_{U_P})_{(g)}$ and Claim 4 we compute

$$\left|(\Gamma_{U_P})_{(g)}/H\right| \leq \prod_{\alpha\in\Psi_{\mathrm{red}}} \left|f_\alpha(U^{g_M}_{(\alpha,0)}) \cap U_{(\alpha,0)}/U_{(\alpha,r)}\right|$$
$$\leq \prod_{\alpha\in\Psi_{\mathrm{red}}} \left|(U_{(\alpha,0)})_{(g_M)}/U_{(\alpha,r)}\right| = \left|(\Gamma_{U_P})_{(g_M)}/H\right|.$$

Finally, we obtain

$$\mu_{U_P}(g) = [\Gamma_{U_P} : (\Gamma_{U_P})_{(g)}] = \frac{|\Gamma_{U_P}/H|}{|(\Gamma_{U_P})_{(g)}/H|}$$
$$\geq \frac{|\Gamma_{U_P}/H|}{|(\Gamma_{U_P})_{(g_M)}/H|} = [\Gamma_{U_P} : (\Gamma_{U_P})_{(g_M)}] = \mu_{U_P}(g_M).$$

□

Corollary 4.8. *The set $\left\{\mu_{U_P}(m) \cdot (m)_{\Gamma_M} \,\middle|\, m \in M\right\}$ is a $\mathbb{Z}$-basis of $\Theta^P_{M,\mathbb{Z}}$ (3.1.5). In particular, the image of $\Theta^P_{M,R}$ equals $H_R(\Gamma_M, M^+)$ whenever $qR = 0$.*

Proof. Proposition 4.7 implies that $\mu_{U_P}(g_M)$ divides $\nu_M(g)\mu_{U_P}(g)$ for all $g \in P$, as all occuring numbers are powers of q. Hence the first assertion. Since $m \in M$ is positive if and only if $\mu_{U_P}(m) = [\Gamma_{U_P} : \Gamma_{U_P} \cap m^{-1}\Gamma_{U_P} m] = 1$ (Corollary 1.63 using $\Gamma_{U_P} = I_{U_P}$), we conclude that the image of $\Theta^P_{M,R}$ equals $H_R(\Gamma_M, M^+)$ provided that $qR = 0$. □

Recall from Definition 1.60 that an element $a \in M^+$ is called *strictly positive* if it lies in the center $Z(M)$ of M and satisfies $\bigcap_{n\in\mathbb{N}} a^n K_{U_P} a^{-n} = \{1\}$ and $\bigcap_{n\in\mathbb{N}} a^{-n} K_{U_{P^{\mathrm{op}}}} a^n = \{1\}$. There always exist strictly positive elements (Remark 1.61).

Proposition 4.9. *Let $a_P \in M^+$ be a strictly positive element and consider the centralizer*

$$C_P^+ := \left\{x \in H_R(\Gamma, P) \,\middle|\, x \cdot (a_P)_\Gamma = (a_P)_\Gamma \cdot x\right\}$$

of $(a_P)_\Gamma$ in $H_R(\Gamma, P)$. Then $\{(m)_\Gamma \mid m \in M^+\}$ is an R-basis for C_P^+. In particular, C_P^+ does not depend on the choice of the strictly positive element a_P. The restriction of Θ^P_M to C_P^+ induces an isomorphism $C_P^+ \cong H_R(\Gamma_M, M^+)$.

Proof. See Lemma 3.14 and Corollary 3.15. The proof applies in this generality, except that Lemma 3.3 needs to be replaced by Corollary 1.63. □

4.3. Construction of Ξ_G^P

Recall the set of affine roots $\Sigma^{\mathrm{aff}} := \Sigma \times \mathbb{Z}$ (Proposition 1.32). The Iwahori-Weyl group W, and by inflation also $W(1)$, acts on Σ^{aff} (1.5.3).

Definition 4.10. For $(\alpha, k) \in \Sigma^{\mathrm{aff}}$ we define

$$q(\alpha, k) := \left|U_{(\alpha,k)}/U_{(\alpha,k+1)}\right| \in q^{\mathbb{N}}. \tag{4.3.1}$$

Lemma 4.11. *(i) For all $(\alpha, k) \in \Sigma^{\mathrm{aff}}$ and all $w \in W$ or $w \in W(1)$ we have*

$$q(\alpha, k) = q\big(w \cdot (\alpha, k)\big).$$

In particular, we have $q(\alpha, k) = q(-\alpha, -k)$ and $q(\alpha, k + 2m) = q(\alpha, k)$ for all $(\alpha, k) \in \Sigma^{\mathrm{aff}}$ and $m \in \mathbb{Z}$.

(ii) For all $(\alpha, k) \in \Sigma^{\mathrm{aff}}$ we have

$$q(\alpha, k) = q(H_{(\alpha,k)}),$$

where $q(H_{(\alpha,k)})$ was defined in Definition 2.24 and (2.2.2).

Proof. (i) Let $(\alpha, k) \in \Sigma^{\mathrm{aff}}$ and $w \in W$ (the case $w \in W(1)$ follows trivially from this case). Since $W \cong \Lambda \rtimes W_0$, there exist $\lambda \in \Lambda$ and $w_0 \in W_0$ with $w = e^{\lambda} w_0$. Let $n \in N$ with $w = nZ_0$. Using (1.5.4) we compute

$$nU_{(\alpha,k)}n^{-1} = U_{w\cdot(\alpha,k)} = U_{(w_0(\alpha),k-\langle w_0(\alpha),\nu(\lambda)\rangle)}.$$

Hence, conjugation by n induces an isomorphism

$$U_{(\alpha,k)}/U_{(\alpha,k+1)} \xrightarrow{\sim} U_{(w_0(\alpha),k-\langle w_0(\alpha),\nu(\lambda)\rangle)}/U_{(w_0(\alpha),k-\langle w_0(\alpha),\nu(\lambda)\rangle+1)}.$$

Comparing cardinalities, this shows $q(\alpha, k) = q\big(w_0(\alpha), k - \langle w_0(\alpha), \nu(\lambda)\rangle\big) = q\big(w \cdot (\alpha, k)\big)$. This proves the first assertion.

Take $m \in \mathbb{Z}$. Recall that $Q(\Sigma^\vee) \subseteq W^{\mathrm{aff}} \subseteq W$ (1.5.2). Hence, taking $\lambda \in \Lambda$ with $\nu(\lambda) = -m\alpha^\vee$ (see also (1.6.5)) and $w_0 = 1$, we obtain

$$q(\alpha, k) = q\big(e^{\lambda} \cdot (\alpha, k)\big) = q(\alpha, k - \langle \alpha, \nu(\lambda)\rangle) = q(\alpha, k + 2m).$$

Finally, taking $w_0 = s_{\alpha,\alpha^\vee}$ and $m = -k$ we obtain

$$q(\alpha, k) = q(w_0 \cdot (\alpha, k)\big) = q(-\alpha, k) = q(-\alpha, -k).$$

(ii) Let $s \in S^{\mathrm{aff}}$ and take $\alpha \in \Phi^+_{\mathrm{red}}$ and $r \in \Gamma_\alpha$ with $H_s = H_{\alpha,r}$. Put $\alpha_s := \alpha$ and $r_s := r$ if $r \in \Gamma'_\alpha$ (1.4.2); otherwise put $\alpha_s := 2\alpha$ and $r_s := 2r \in \Gamma_{2\alpha}$. By definition we have $q_s = |U_{\alpha_s,r_s}/U_{\alpha_s,r_s+}|$ (2.2.2). Write $\beta := \varepsilon_\alpha \alpha \in \Sigma$ (see Remark 1.33, (a)) and $k := \varepsilon_\alpha r \in \mathbb{Z}$. Then $H_s = H_{\alpha,r} = H_{2\alpha,2r} = H_{(\beta,k)}$. If $r \notin \Gamma'_\alpha$, we have an isomorphism $U_{2\alpha,2r}/U_{2\alpha,2r+} \cong U_{\alpha,r}/U_{\alpha,r+}$ by Lemma 1.25, (ii). In any case we compute

$$\begin{aligned} q(\beta,k) &= \left|U_{(\beta,k)}/U_{(\beta,k+1)}\right| = \left|U_{\alpha,r}/U_{\alpha,r+}\right| \\ &= \left|U_{\alpha_s,r_s}/U_{\alpha_s,r_s+}\right| = q_s = q(H_{(\beta,k)}). \end{aligned}$$

By (i) (and Definition 2.24) we conclude that $q(\alpha,k) = q(H_{(\alpha,k)})$ for all $(\alpha,k) \in \Sigma^{\mathrm{aff}}$.

□

Notation 4.12. Recall that $\mu_{U_P} : M \to q^{\mathbb{Z}_{\geq 0}}$ is constant on double cosets with respect to the Iwahori subgroup I_M (Remark 2.4), and that we have a natural bijection $W_M \cong I_M\backslash M/I_M$ (Proposition 1.46). Given any $m \in M$, we write

$$\mu_{U_P}(w) := \mu_{U_P}(m),$$

where $w \in W_M$ is the element representing the double coset $I_M m I_M$. Since the map $\mu_{U_P} : M \to q^{\mathbb{Z}_{\geq 0}}$ is even constant on double cosets with respect to K_M, and K_M contains representatives of $W_{0,M}$ (Remark 1.45), we have $\mu_{U_P}(ww_0) = \mu_{U_P}(w_0 w) = \mu_{U_P}(w)$ for all $w_0 \in W_{0,M}$. In particular, we have $\mu_{U_P}(m) = \mu_{U_P}(\lambda)$ for some $\lambda \in \Lambda$. By inflation, μ_{U_P} is also defined on $W_M(1)$.

Lemma 4.13. *Let $w_0 \in W_{0,M}$ and $x, y \in \Lambda$ such that the following conditions are satisfied:*

- $\langle \alpha, \nu(y)\rangle = 0$ *for all* $\alpha \in \Sigma_M$*;*
- $\langle \alpha, \nu(y)\rangle < 0$ *for all* $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$*;*
- $\langle \alpha, \nu(x+y)\rangle \leq 0$ *for all* $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$*.*

Then we have $q_{e^y, e^x w_0} = \mu_{U_P}(e^x w_0)$*.*

Proof. Step 1: We prove $q_{e^y,e^x w_0} = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle>0}} |U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(x)\rangle)}|$. By Lemma 2.26, (iii) we have $q_{e^y,e^x w_0} = \prod_{H\in\mathfrak{H}_{e^y}\cap e^y\mathfrak{H}_{e^x w_0}} q(H)$, where $\mathfrak{H}_w$, for $w \in W$, denotes the set of hyperplanes separating $w\mathfrak{C}$ and $\mathfrak{C}$. By Lemma 1.51 we have:

- $\ell_\alpha(e^y) = \langle \alpha, \nu(y)\rangle$ for all $\alpha \in \Sigma^+$;
- $\ell_\alpha(e^x w_0) = \langle \alpha, \nu(x)\rangle$ for all $\alpha \in w_0(\Sigma^+)$;
- $\ell_\alpha(e^x w_0) = \langle \alpha, \nu(x)\rangle - 1$ for all $\alpha \in w_0(\Sigma^-)$.

Recall that $w_0(\Sigma^+ \smallsetminus \Sigma_M) = \Sigma^+ \smallsetminus \Sigma_M$ (see the proof of Proposition 1.62). Using Lemma 1.49 we have

$$\begin{aligned} \mathfrak{H}_{e^y} &= \left\{H_{(\alpha,k)} \,\middle|\, \alpha \in \Sigma^+ \smallsetminus \Sigma_M \text{ and } k \in \{0,1,\dots,-\langle\alpha,\nu(y)\rangle - 1\}\right\} \\ \mathfrak{H}_{e^x w_0} &= \left\{H_{(\alpha,k)} \,\middle|\, \alpha \in \Sigma^+ \smallsetminus \Sigma_M \text{ with } \langle\alpha,\nu(x)\rangle > 0 \text{ and } k \in \{-\langle\alpha,\nu(x)\rangle,\dots,-1\}\right\} \\ &\quad \cup \left\{H_{(\alpha,k)} \,\middle|\, \begin{array}{l} \alpha \in \Sigma^+ \smallsetminus \Sigma_M \text{ with } \langle\alpha,\nu(x)\rangle < 0 \\ \text{and } k \in \{0,\dots,-\langle\alpha,\nu(x)\rangle - 1\} \end{array}\right\} \\ &\quad \cup \left\{H_{(\alpha,k)} \,\middle|\, (\alpha,k) \in \Sigma_M^+ \times \mathbb{Z} \text{ with } H_{(\alpha,k)} \in \mathfrak{H}_{e^x w_0}\right\}. \end{aligned}$$

Notice that $e^y H_{(\alpha,k)} = H_{(\alpha,k-\langle\alpha,\nu(y)\rangle)}$ for all $(\alpha,k) \in \Sigma^{\mathrm{aff}}$. It follows that

$$\mathfrak{H}_{e^y} \cap e^y \mathfrak{H}_{e^x w_0} = \left\{ H_{(\alpha,k)} \;\middle|\; \begin{array}{l} \alpha \in \Sigma^+ \smallsetminus \Sigma_M \text{ with } \langle \alpha, \nu(x)\rangle > 0 \text{ and} \\ k \in \{-\langle\alpha,\nu(x+y)\rangle, \dots, -\langle \alpha, \nu(y)\rangle - 1\} \end{array} \right\}.$$

Thus, using Lemma 4.11 we compute

$$\begin{aligned} q_{e^y, e^x w_0} &= \prod_{H \in \mathfrak{H}_{e^y} \cap e^y \mathfrak{H}_{e^x w_0}} q(H) = \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \prod_{k=0}^{\langle\alpha,\nu(x)\rangle - 1} q\big(\alpha, k - \langle\alpha,\nu(x+y)\rangle\big) \\ &= \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \prod_{k=0}^{\langle\alpha,\nu(x)\rangle - 1} q\big(e^{x+y} \cdot (\alpha,k)\big) = \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \prod_{k=0}^{\langle\alpha,\nu(x)\rangle - 1} q(\alpha,k) \\ &= \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \prod_{k=0}^{\langle\alpha,\nu(x)\rangle - 1} \big|U_{(\alpha,k)}/U_{(\alpha,k+1)}\big| = \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \big|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(x)\rangle)}\big|. \end{aligned}$$

Step 2: We show $\mu_{U_P}(e^x w_0) = \mu_{U_P}(e^x) = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle>0}} \big|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(x)\rangle)}\big|$. The first equality is clear. We prove the second one. By definition there exists $m \in Z$ with $mZ_0 = x \in \Lambda$. We have a homeomorphism $\prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,0)} \cong I_{U_P}$ (which is induced by the multiplication map). Therefore,

$$\begin{aligned} I_{U_P} \cap m^{-1} I_{U_P} m &\cong \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} \big(U_{(\alpha,0)} \cap m^{-1} U_{(\alpha,0)} m\big) = \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} \big(U_{(\alpha,0)} \cap U_{(\alpha,\langle\alpha,\nu(x)\rangle)}\big) \\ &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle \le 0}} U_{(\alpha,0)} \times \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} U_{(\alpha,\langle\alpha,\nu(x)\rangle)}. \end{aligned}$$

Take any $k \in \mathbb{Z}_{\ge 0}$ with $k \ge \max\{\langle\alpha,\nu(x)\rangle \mid \alpha \in \Sigma^+ \smallsetminus \Sigma_M\}$ and consider the subgroup H generated by $\bigcup_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,k)}$. Then H is normal and by Lemma 1.6 the multiplication map induces a homeomorphism $\prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} U_{(\alpha,k)} \cong H$. By construction H is contained in $(I_{U_P})_{(m)}$. We have

$$\big|I_{U_P}/H\big| = \prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_M} \big|U_{(\alpha,0)}/U_{(\alpha,k)}\big|$$

and

$$\big|(I_{U_P})_{(m)}/H\big| = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle \le 0}} \big|U_{(\alpha,0)}/U_{(\alpha,k)}\big| \cdot \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \big|U_{(\alpha,\langle\alpha,\nu(x)\rangle)}/U_{(\alpha,k)}\big|.$$

Hence, it follows that

$$\begin{aligned} \mu_{U_P}(e^x) = [I_{U_P} : (I_{U_P})_{(m)}] = \frac{|I_{U_P}/H|}{|(I_{U_P})_{(m)}/H|} &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \frac{|U_{(\alpha,0)}/U_{(\alpha,k)}|}{|U_{(\alpha,\langle\alpha,\nu(x)\rangle)}/U_{(\alpha,k)}|} \\ &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M \\ \langle\alpha,\nu(x)\rangle > 0}} \big|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(x)\rangle)}\big|. \end{aligned}$$

Combining Steps 1 and 2 yields the assertion. $\square$

Proposition 4.14. *If p is not a zero-divisor in R, then the R-algebra homomorphism $\theta^+\colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ (2.3.1) extends uniquely to an R-algebra homomorphism $\widetilde{\theta}^+\colon \operatorname{Im}\Theta_M^P \to \mathcal{H}_R(G)$, and $\operatorname{Im}\Theta_M^P$ is the maximal subalgebra of $\mathcal{H}_R(M)$ with this property.*

For arbitrary R, we obtain, in particular, an R-algebra homomorphism

$$\Xi_G^P := \Xi_{G,R}^P := \mathrm{id}_R \otimes (\widetilde{\theta}^+ \circ \Theta_{M,\mathbb{Z}}^P)\colon H_R\big(I_P(1), P\big) \longrightarrow \mathcal{H}_R(G).$$

Proof. We assume first that p is not a zero-divisor in R. The inclusion $R \hookrightarrow R[p^{-1}]$ induces inclusions $\mathcal{H}_R(M) \subseteq \mathcal{H}_{R[p^{-1}]}(M)$ and $\mathcal{H}_R(G) \subseteq \mathcal{H}_{R[p^{-1}]}(G)$. Let $a \in M^+$ be a strictly positive element (it always exists by Remark 1.61). Then T_a is invertible in $\mathcal{H}_{R[p^{-1}]}(G)$. By Proposition 2.29 the map $\theta^+\colon \mathcal{H}_{R[p^{-1}]}(M^+) \to \mathcal{H}_{R[p^{-1}]}(G)$ extends uniquely to an $R[p^{-1}]$-algebra homomorphism $\widetilde{\theta}^+\colon \mathcal{H}_{R[p^{-1}]}(M) \to \mathcal{H}_{R[p^{-1}]}(G)$ defined as follows: given $m \in M$, there exists $n \in \mathbb{N}$ such that $a^n m$ lies in M^+. Now, $\widetilde{\theta}^+(T_m^M)$ is given by

$$\widetilde{\theta}^+(T_m^M) := T_a^{-n} \cdot \theta^+\big(T_{a^n m}^M\big) = T_a^{-n} T_{a^n m}.$$

Claim. *We have $(\widetilde{\theta}^+)^{-1}\big(\mathcal{H}_R(G)\big) = \operatorname{Im}\Theta_M^P \subseteq \mathcal{H}_R(M)$. In particular, θ^+ extends uniquely to an R-algebra homomorphism $\operatorname{Im}\Theta_M^P \to \mathcal{H}_R(G)$, and $\operatorname{Im}\Theta_M^P$ is maximal with this property.*

Proof of the claim. By Corollary 4.8 the set $\big\{\mu_{U_P}(m) \cdot T_m^M \,\big|\, m \in M\big\}$ is an R-basis of $\operatorname{Im}\Theta_M^P$. As a is a strictly positive element, we have $a \in Z$ and the double coset $I_M a I_M$ is represented by some $\lambda \in \Lambda$ with $\langle \alpha, \nu(\lambda)\rangle = 0$ for all $\alpha \in \Sigma_M$ and $\langle \alpha, \nu(\lambda)\rangle < 0$ for all $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$ (see Proposition 1.62, (iii)).

Given $m \in M$, there exists $n \in \mathbb{N}$ with $a^n m \in M^+$. Let $v \in \Lambda(1)$ be the element representing the double coset $I_M(1) a^n I_M(1)$. If $w \in W_M(1)$ represents the double coset $I_M(1) m I_M(1)$, then vw represents $I_M(1) a^n m I_M(1)$, since a is central. Clearly, $n\lambda$ is the image of v in W_M. Denote by $e^x w_0 \in \Lambda \rtimes W_{0,M}$ the image of w in W_M. As $a^n m$ is positive, it follows from Proposition 1.62 that $\langle \alpha, \nu(n\lambda + x)\rangle \leq 0$ for all $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$. Hence, Lemma 4.13 is applicable and shows $q_{v,w} = q_{e^{n\lambda}, e^x w_0} = \mu_{U_P}(e^x w_0) = \mu_{U_P}(m)$. Applying the Fundamental Lemma (Corollary 2.23) yields

$$\begin{aligned}\widetilde{\theta}^+\big(\mu_{U_P}(m) \cdot T_m^M\big) &= \mu_{U_P}(m) \cdot T_a^{-n} T_{a^n m} \\ &= q_{v,w} \cdot T_v^{-1} \cdot T_{vw} = T_w + \sum_{w' < w} \lambda_{w'} T_{w'} \in \mathcal{H}_R(G) \qquad (4.3.2)\end{aligned}$$

for some $\lambda_{w'} \in \mathbb{Z}$ (viewed as elements of R), where "$<$" is the Bruhat order in $W(1)$. We conclude that $\operatorname{Im}\Theta_M^P$ is contained in $(\widetilde{\theta}^+)^{-1}\big(\mathcal{H}_R(G)\big)$.

Conversely, let $X := \sum_{i=1}^k x_i \cdot T_{w_i}^M \in \mathcal{H}_{R[p^{-1}]}(M)$ be an element with $x_i \in R[p^{-1}] \smallsetminus \{0\}$ for all $1 \leq i \leq k$ and with $\widetilde{\theta}^+(X) \in \mathcal{H}_R(G)$. We prove $X \in \operatorname{Im}\Theta_M^P$ by induction on k. The case $k = 0$ is trivial. Assume now that $k > 0$ and the claim is true for $k-1$. Rearranging if necessary, we may assume that $w_k \in W_M(1)$ is maximal among $\{w_1, \dots, w_k\}$ with respect to the Bruhat order in $W(1)$. Take $m \in M$ with $T_m^M = T_{w_k}^M$. Let $n \in \mathbb{N}$ such that $T_{a^n}^M X \in \mathcal{H}_{R[p^{-1}]}(M^+)$ and let $v \in \Lambda(1)$ be the element representing $I_M(1) a^n I_M(1)$. We have $T^M_{a^n}X = \sum_{i=1}^k x_i \cdot T^M_{vw_i}$ and therefore,

$$\widetilde{\theta}^+(X) = \sum_{i=1}^k x_i \cdot T_v^{-1} T_{vw_i} = \sum_{i=1}^k x_i q_{v,w_i}^{-1} \cdot \left(T_{w_i} + \sum_{w_i' < w_i} \lambda_{w_i',i} T_{w_i'} \right).$$

By maximality of w_k the coefficient before T_{w_k} is $x_k \cdot q_{v,w_k}^{-1}$. Since $q_{v,w_k} = \mu_{U_P}(m)$ and $\widetilde{\theta}^+(X) \in \mathcal{H}_R(G)$, we have $x_k \cdot \mu_{U_P}(m)^{-1} \in R$; in particular, $x_k \in R$ and $\mu_{U_P}(m)$ divides x_k in R. This shows $x_k \cdot T^M_{w_k} \in \operatorname{Im} \Theta^P_M$. By the induction hypothesis we have $X - x_k T^M_{w_k} \in \operatorname{Im} \Theta^P_M$, and hence $X \in \operatorname{Im} \Theta^P_M$. This proves the equality.

The uniqueness of $\operatorname{Im} \Theta^P_M \to \mathcal{H}_R(G)$ is a consequence of the uniqueness of $\widetilde{\theta}^+$: any such homomorphism induces a homomorphism $R[p^{-1}] \otimes_R \operatorname{Im} \Theta^P_M \to \mathcal{H}_{R[p^{-1}]}(G)$, which has to coincide with $\widetilde{\theta}^+|_{R[p^{-1}] \otimes_R \operatorname{Im} \Theta^P_M}$ (notice that $R[p^{-1}]$ is a flat R-module). The claim is proved. □

This proves the first assertion of the proposition. The last assertion follows via extension of scalars, using $H_R(I_P(1), P) = R \otimes_{\mathbb{Z}} H_{\mathbb{Z}}(I_P(1), P)$ and $\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G)$. □

4.4. On the properties of $\mu_{U_P}(w)$

Recall Notation 4.12, where we defined $\mu_{U_P}(w)$ for $w \in W_M$ or $w \in W_M(1)$. Also recall that $\mu_{U_P} : W_M \to q^{\mathbb{Z}_{\geq 0}}$ is constant on double cosets with respect to $W_{0,M}$. All statements in this section about $\mu_{U_P}(w)$ for $w \in W_M$ are trivially also true for $w \in W_M(1)$. We therefore refrain from mentioning this case each time.

We consider the opposite parabolic subgroup $\mathbf{P}^{\mathrm{op}}$ with unipotent radical $\mathbf{U}_{P^{\mathrm{op}}} = \prod_{\alpha \in \Sigma^- \smallsetminus \Sigma_M} \mathbf{U}_\alpha$, where we have identified $\Sigma^- \smallsetminus \Sigma_M$ with the reduced roots in $\Phi^- \smallsetminus \Phi_M$. We have a decomposition $I_{P^{\mathrm{op}}} = I_M I_{U_{P^{\mathrm{op}}}}$ with I_M normalizing $I_{U_{P^{\mathrm{op}}}}$. We put $\mu_{U_{P^{\mathrm{op}}}}(g) := [I_{U_{P^{\mathrm{op}}}} : I_{U_{P^{\mathrm{op}}}} \cap g^{-1} I_{U_{P^{\mathrm{op}}}} g]$ for $g \in P^{\mathrm{op}}$. Analogously to the above we define $\mu_{U_{P^{\mathrm{op}}}} : W_M \to q^{\mathbb{Z}_{\geq 0}}$. The relation between μ_{U_P} and $\mu_{U_{P^{\mathrm{op}}}}$ is given by the following lemma.

Lemma 4.15. *For each $w \in W_M$ we have $\mu_{U_{P^{\mathrm{op}}}}(w) = \mu_{U_P}(w^{-1})$.*

Proof. Write $w = e^\lambda w_0$ with $\lambda \in \Lambda$ and $w_0 \in W_{0,M}$. As both μ_{U_P} and $\mu_{U_{P^{\mathrm{op}}}}$ are constant on double cosets with respect to K_M, we have $\mu_{U_P}(w^{-1}) = \mu_{U_P}(w_0^{-1} e^{-\lambda}) = \mu_{U_P}(-\lambda)$ and $\mu_{U_{P^{\mathrm{op}}}}(e^\lambda w_0) = \mu_{U_{P^{\mathrm{op}}}}(\lambda)$. It therefore suffices to show $\mu_{U_{P^{\mathrm{op}}}}(\lambda) = \mu_{U_P}(-\lambda)$.

In Step 2 in the proof of Lemma 4.13 we have seen

$$\mu_{U_P}(-\lambda) = \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle \alpha, \nu(-\lambda) \rangle > 0}} \left| U_{(\alpha,0)} / U_{(\alpha, \langle \alpha, \nu(-\lambda) \rangle)} \right| = \prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma_M \\ \langle \alpha, \nu(-\lambda) \rangle > 0}} \prod_{k=0}^{\langle \alpha, \nu(-\lambda) \rangle - 1} q(\alpha, k).$$

An analogous argument shows

$$\mu_{U_{P^{\mathrm{op}}}}(\lambda) = \prod_{\substack{\alpha \in \Sigma^- \smallsetminus \Sigma_M \\ \langle \alpha, \nu(\lambda) \rangle > 0}} \left| U_{(\alpha,1)} / U_{(\alpha, \langle \alpha, \nu(\lambda) \rangle + 1)} \right| = \prod_{\substack{\alpha \in \Sigma^- \smallsetminus \Sigma_M \\ \langle \alpha, \nu(\lambda) \rangle > 0}} \prod_{k=1}^{\langle \alpha, \nu(\lambda) \rangle} q(\alpha, k).$$

Notice that $\Sigma^- \setminus \Sigma_M = -(\Sigma^+ \setminus \Sigma_M)$ and $\langle -\alpha, \nu(-\lambda)\rangle = \langle \alpha, \nu(\lambda)\rangle$ for $\alpha \in \Sigma^- \setminus \Sigma_M$. By Lemma 4.11 (i) we have $q(\alpha,k) = q(-\alpha,k)$ for all $(\alpha,k) \in \Sigma^{\mathrm{aff}}$ and $q(-\alpha,0) = q\big(\alpha,\langle \alpha,\nu(\lambda)\rangle\big)$ for $\alpha \in \Sigma^- \setminus \Sigma_M$. Put together, this shows $\mu_{U_P}(-\lambda) = \mu_{U_{P^{\mathrm{op}}}}(\lambda)$. □

Recall the numbers q_w and $q_{v,w}$ for $v, w \in W$ that were studied in section 2.2, especially Lemmas 2.25 and 2.26. These numbers are intimately related to the length function on W. If for instance $\mathbf{G}$ is F-split, we have $q_w = q^{\ell(w)}$ for all $w \in W$. In general we have $q_{v,w} = 1$ if and only if $\ell(vw) = \ell(v) + \ell(w)$. As $\mathbf{M}$ is a connected reductive group, the analogous numbers $q_{M,w}$ and $q_{M,v,w}$ are defined for $v, w \in W_M$. Recall that in general $S_M^{\mathrm{aff}} \not\subseteq S^{\mathrm{aff}}$ and hence the length function ℓ_M on W_M is not the restriction of the length function ℓ on W to W_M. Nevertheless, they are related to each other and the next result quantifies this.

Proposition 4.16. *Let $w, v \in W_M$. Then we have:*

(i) $q_w = \mu_{U_P}(w)\mu_{U_P}(w^{-1}) \cdot q_{M,w}$;

(ii) $\mu_{U_P}(vw) \le \mu_{U_P}(v)\mu_{U_P}(w)$ *and* $q_{v,w} = \frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)} \cdot q_{M,v,w}$;

(iii) $\mu_{U_P}(v) \le \mu_{U_P}(w)$ *whenever* $v \le_M w$, *where "$\le_M$" is the Bruhat order on* W_M.

Proof. (i) By Lemma 1.56 there is a canonical affine N_M-equivariant surjection $p_M : \mathscr{A} \twoheadrightarrow \mathscr{A}_M$ such that taking preimages induces an injection $\mathfrak{H}_M \hookrightarrow \mathfrak{H}$ (which we view as an inclusion), and with $\mathfrak{C} \subseteq p_M^{-1}(\mathfrak{C}_M)$ (here, $\mathscr{A}_M$, $\mathfrak{H}_M$ and $\mathfrak{C}_M$ denote the apartment of M, its set of hyperplanes, and the fundamental alcove, respectively). Notice that $H \in \mathfrak{H}_M$ separates $\mathfrak{C}$ and $w\mathfrak{C}$ (for $w \in W_M$) if and only if it separates $\mathfrak{C}_M$ and $w\mathfrak{C}_M$. Hence, if $\mathfrak{H}_{M,w}$ denotes the set of hyperplanes in $\mathfrak{H}_M$ separating $\mathfrak{C}_M$ and $w\mathfrak{C}_M$, we have $\mathfrak{H}_{M,w} = \mathfrak{H}_w \cap \mathfrak{H}_M$. Also notice that $\mathfrak{H}_M = \big\{H_{(\alpha,k)} \in \mathfrak{H} \,\big|\, (\alpha,k) \in \Sigma_M^{\mathrm{aff}} = \Sigma_M \times \mathbb{Z}\big\}$.

Given $w = e^{\lambda} w_0 \in W_M$ with $\lambda \in \Lambda$ and $w_0 \in W_{0,M}$, we computed in Step 1 in the proof of Lemma 4.13 that $\mathfrak{H}_w = \mathfrak{H}_w^+ \sqcup \mathfrak{H}_w^- \sqcup \mathfrak{H}_{M,w}$, where

$$\mathfrak{H}_w^+ := \left\{ H_{(\alpha,k)} \,\middle|\, \begin{array}{l} \alpha \in \Sigma^+ \setminus \Sigma_M \text{ with } \langle \alpha, \nu(\lambda)\rangle > 0 \\ \text{and } k \in \{-\langle \alpha,\nu(\lambda)\rangle, \dots, -1\} \end{array} \right\},$$
$$\mathfrak{H}_w^- := \left\{ H_{(\alpha,k)} \,\middle|\, \begin{array}{l} \alpha \in \Sigma^+ \setminus \Sigma_M \text{ with } \langle \alpha, \nu(\lambda)\rangle < 0 \\ \text{and } k \in \{0, \dots, -\langle \alpha,\nu(\lambda)\rangle - 1\} \end{array} \right\}.$$

By Lemma 2.25 we have $q_w = \prod_{H \in \mathfrak{H}_w} q(H)$ and $q_{M,w} = \prod_{H \in \mathfrak{H}_{M,w}} q(H)$. Moreover, we compute, using Lemma 4.11 and the proof of Lemma 4.15,

$$\begin{aligned} \prod_{H \in \mathfrak{H}_w^+} q(H) &= \prod_{\substack{\alpha \in \Sigma^+ \setminus \Sigma_M \\ \langle \alpha,\nu(\lambda)\rangle > 0}} \prod_{k=0}^{\langle \alpha,\nu(\lambda)\rangle - 1} q\big(\alpha, k - \langle \alpha,\nu(\lambda)\rangle\big) \\ &= \prod_{\substack{\alpha \in \Sigma^+ \setminus \Sigma_M \\ \langle \alpha,\nu(\lambda)\rangle > 0}} \prod_{k=0}^{\langle \alpha,\nu(\lambda)\rangle - 1} q(\alpha,k) = \mu_{U_P}(\lambda) = \mu_{U_P}(w), \end{aligned}$$

and

$$\prod_{H\in\mathfrak{H}_w^-} q(H) = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M\\ \langle\alpha,\nu(\lambda)\rangle<0}} \prod_{k=0}^{-\langle\alpha,\nu(\lambda)\rangle-1} q(\alpha,k) = \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M\\ \langle\alpha,\nu(-\lambda)\rangle>0}} \prod_{k=0}^{\langle\alpha,\nu(-\lambda)\rangle-1} q(\alpha,k)$$
$$= \mu_{U_P}(-\lambda) = \mu_{U_P}(w^{-1}).$$

Thus, we obtain

$$q_w = \prod_{H\in\mathfrak{H}_w^+} q(H)\cdot \prod_{H\in\mathfrak{H}_w^-} q(H)\cdot \prod_{H\in\mathfrak{H}_{M,w}} q(H) = \mu_{U_P}(w)\mu_{U_P}(w^{-1})\cdot q_{M,w}.$$

(ii) By Remark 2.4 (b) the map $\delta_P\colon P\longrightarrow \mathbb{Q}_{>0}^\times$, $g\mapsto \delta_P(g) := \mu_{U_P}(g)\cdot\mu_{U_P}(g^{-1})^{-1}$ is a group homomorphism. Its restriction to N_M factors through W_M. Therefore, the map $\delta_M\colon W_M\longrightarrow \mathbb{Q}_{>0}^\times$, $w\mapsto \mu_{U_P}(w)\cdot\mu_{U_P}(w^{-1})^{-1}$ is a group homomorphism. Given $v,w\in W_M$, this implies $\frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)} = \frac{\mu_{U_P}(v^{-1})\mu_{U_P}(w^{-1})}{\mu_{U_P}(w^{-1}v^{-1})}$. Applying (i) we therefore compute

$$q_{v,w}^2 = \frac{q_v q_w}{q_{vw}} = \frac{\mu_{U_P}(v)\mu_{U_P}(v^{-1})\cdot\mu_{U_P}(w)\mu_{U_P}(w^{-1})}{\mu_{U_P}(vw)\mu_{U_P}(w^{-1}v^{-1})}\cdot\frac{q_{M,v}q_{M,w}}{q_{M,vw}}$$
$$= \left(\frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)}\cdot q_{M,v,w}\right)^2.$$

Taking roots, the second statement follows. Since $q_{v,w} = \prod_{H\in\mathfrak{H}_v\cap v\mathfrak{H}_w} q(H)$, $q_{M,v,w} = \prod_{H\in\mathfrak{H}_{M,v}\cap v\mathfrak{H}_{M,w}} q(H)$, and $\mathfrak{H}_{M,v}\cap v\mathfrak{H}_{M,w}\subseteq \mathfrak{H}_v\cap v\mathfrak{H}_w$, we see that $q_{M,v,w}$ always divides $q_{v,w}$. This shows $\frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)}\in q^{\mathbb{Z}_{\geq 0}}$ and hence the first statement.

(iii) Recall that, given $v,w\in W_M^{\mathrm{aff}}$ and $u,u'\in\Omega_M$, we have $vu\leq_M wu'$ if and only if $v\leq_M w$ and $u=u'$. Let $v,w\in W_M$ be such that $v\leq_M w$. By Proposition 1.37 and the subsequent remark we may reduce to the case $v=tw$ for some $t\in S(\mathfrak{H}_M)$ with $\ell_M(tw)=\ell_M(w)-1$. We then have to show $\mu_{U_P}(tw)\leq\mu_{U_P}(w)$. This is done in three steps.

Step 1: We show $\mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big)\leq\mu_{U_P}(w)\mu_{U_P}(w^{-1})$. The hypothesis $\ell_M(tw)=\ell_M(w)-1$ means that $H_t\in\mathfrak{H}_{M,w}\smallsetminus\mathfrak{H}_{M,tw}$. This also means $H_t\in\mathfrak{H}_w\smallsetminus\mathfrak{H}_{tw}$ (see the proof of (i)). Write $w=s_1\cdots s_n u$ with $s_i\in S^{\mathrm{aff}}$ and $u\in\Omega$ such that $n=\ell(w)$, and consider the gallery

$$\Gamma_w := (\mathfrak{C}, s_1\mathfrak{C}, s_1s_2\mathfrak{C},\dots,s_1s_2\cdots s_n\mathfrak{C})$$

in $\mathscr{A}$. Let i be the index such that Γ_w crosses H_t when going from $s_1\cdots s_{i-1}\mathfrak{C}$ to $s_1\cdots s_i\mathfrak{C}$. Then $t=(s_1\cdots s_{i-1})s_i(s_1\cdots s_{i-1})^{-1}$ and $tw=s_1\cdots\widehat{s_i}\cdots s_n u$. We obtain a (possibly not minimal) gallery connecting $\mathfrak{C}$ and $tw\mathfrak{C}$

$$\Gamma_{tw} := (\mathfrak{C}, s_1\mathfrak{C},\dots,s_1\cdots s_{i-1}\mathfrak{C}, ts_1\cdots s_{i+1}\mathfrak{C},\dots,ts_1\cdots s_n\mathfrak{C})$$

of length $\ell(w)-1$. Note that this gallery does not cross H_t. We obtain Γ_{tw} by folding Γ_w along H_t and deleting one of the alcoves $s_1\cdots s_{i-1}\mathfrak{C}$ appearing twice in succession in the resulting gallery. Clearly, Γ_{tw} crosses all the hyperplanes inside $\mathfrak{H}_{tw}$. We compute

$$\begin{aligned}
\mu_{U_P}(w)\mu_{U_P}(w^{-1})q_{M,w} = q_w &= \prod_{j=1}^{n} q(s_1\cdots s_{j-1}H_{s_j}) \\
&= \prod_{j=1}^{i-1} q(s_1\cdots s_{j-1}H_{s_j})\cdot \prod_{j=i+1}^{n} q(ts_1\cdots s_{j-1}H_{s_j})\cdot q(H_t) \\
&= \prod_{\substack{\text{hyperplanes } H\\ \text{crossed by } \Gamma_{tw}}} q(H)\cdot q(H_t) \\
&\geq q_{tw}\cdot q(H_t) \\
&= \mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big)\cdot q_{M,tw}\cdot q(H_t),
\end{aligned}$$

where the hyperplanes crossed by Γ_{tw} are counted with multiplicity. If we write $w = s'_1\cdots s'_{\ell_M(w)}u'$ with $s'_i\in S_M^{\mathrm{aff}}$ and $u'\in\Omega_M$, then we have $tw = s'_1\cdots\widehat{s'_j}\cdots s'_{\ell_M(w)}u'$ for some $1\leq j\leq \ell_M(w)$. Equivalently, it holds $t = (s'_1\cdots s'_{j-1})s'_j(s'_1\cdots s'_{j-1})^{-1}$, and hence $H_t = s'_1\cdots s'_{j-1}H_{s'_j}$. Thus, recalling $\ell_M(tw)=\ell_M(w)-1$, we compute

$$\begin{aligned}
q_{M,w} &= \prod_{k=1}^{\ell_M(w)} q(s'_1\cdots s'_{k-1}H_{s'_k}) \\
&= \prod_{k=1}^{j-1} q(s'_1\cdots s'_{k-1}H_{s'_k})\cdot \prod_{k=j+1}^{\ell_M(w)} q\big(s'_1\cdots\widehat{s'_j}\cdots s'_{k-1}H_{s'_k}\big)\cdot q(H_t) \\
&= q_{M,tw}\cdot q(H_t).
\end{aligned}$$

Put together this shows $\mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big)\leq \mu_{U_P}(w)\mu_{U_P}(w^{-1})$.

Step 2: We show $\delta_M(tw)=\delta_M(w)$. Since δ_M is multiplicative, we have $\delta_M(tw)=\delta_M(t)\delta_M(w)$. But $t^{-1}=t$ implies $\delta_M(t)=1$, whence the claim.

Step 3: We show $\mu_{U_P}(tw)\leq\mu_{U_P}(w)$. Combining Steps 2 and 3 we compute

$$\begin{aligned}
\mu_{U_P}(tw)^2 &= \mu_{U_P}(tw)\mu_{U_P}\big((tw)^{-1}\big)\cdot\delta_M(tw) \\
&\leq \mu_{U_P}(w)\mu_{U_P}(w^{-1})\cdot\delta_M(w) = \mu_{U_P}(w)^2.
\end{aligned}$$

Taking roots, the statement follows. □

Remark 4.17. (a) The argument in Step 2 of the proof of Proposition 4.16 (iii) shows that the character $\delta_M: W_M\to\mathbb{Q}^\times_{>0}$, $w\mapsto \mu_{U_P}(w)\cdot\mu_{U_P}(w^{-1})^{-1}$ is trivial on W_M^{aff}, and hence factors through Ω_M.

(b) Let o be an orientation of $(\mathcal{A}_M, \mathfrak{H}_M)$ and consider the basis $\big(E_o(w)\big)_{w\in W_M(1)}$ of $\mathcal{H}_{\mathbb{Z}}(M)$. By (2.2.7) we have $E_o(w) = T_w^M + \sum_{v<_M w} \lambda_v T_v^M$ inside $\mathcal{H}_{\mathbb{Z}}(M)$. Proposition 4.16 (iii) implies $\mu_{U_P}(w)E_o(w) \in \operatorname{Im}(\Theta^P_{M,\mathbb{Z}})$. We conclude that $\big(\mu_{U_P}(w)E_o(w)\big)_{w\in W_M(1)}$ is a $\mathbb{Z}$-basis of $\operatorname{Im}(\Theta^P_{M,\mathbb{Z}})$.

(c) In the proof of Proposition 4.16 we have used parts of the proof of Lemma 4.13, but not the lemma itself. In fact, Lemma 4.13 is a consequence of Proposition 4.16 (ii): an element $y \in \Lambda$ with $\langle \alpha, \nu(y)\rangle = 0$ for all $\alpha \in \Sigma_M$ fixes $\mathfrak{C}_M$, and thus lies in Ω_M. Hence, given $w_0 \in W_{0,M}$, $x, y \in \Lambda$ satisfying the hypotheses of Lemma 4.13, we have $q_{M,e^y,e^x w_0} = 1$ (since $\ell_M(e^{y+x}w_0) = \ell_M(e^x w_0) = \ell_M(e^y) + \ell_M(e^x w_0)$) as well as $\mu_{U_P}(e^y) = \mu_{U_P}(e^{y+x}w_0) = 1$. Applying (ii) in the above proposition, we obtain $q_{e^y,e^x w_0} = \mu_{U_P}(e^x w_0)$.

Proposition 4.16 allows us to reprove some well-known results on positive (resp. negative) elements.

Corollary 4.18. *(i) Let either $v, w \in W_{M^+}$ or $v, w \in W_{M^-} := (W_{M^+})^{-1}$. We have $q_{v,w} = q_{M,v,w}$ and in particular $\ell_M(v) + \ell_M(w) - \ell_M(vw) = \ell(v) + \ell(w) - \ell(vw)$.*

(ii) Let $w \in W_{M^+}$ (resp. $w \in W_{M^-}$) and $v \in W_M$ with $v \leq_M w$. Then we have $v \in W_{M^+}$ (resp. $v \in W_{M^-}$).

Proof. (i) Confer [Abe16a, Lem. 4.5]. If $v, w \in W_{M^+}$, we have also $vw \in W_{M^+}$. From Corollary 1.63 it follows that $\mu_{U_P}(v) = \mu_{U_P}(w) = \mu_{U_P}(vw) = 1$. Hence, Proposition 4.16 (ii) shows $q_{v,w} = q_{M,v,w}$. If $v, w \in W_{M^-}$, we have $v^{-1}, w^{-1}, (vw)^{-1} \in W_{M^+}$ and therefore $\frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)} = \frac{\mu_{U_P}(v^{-1})\mu_{U_P}(w^{-1})}{\mu_{U_P}((vw)^{-1})} = 1$. Again we conclude $q_{v,w} = q_{M,v,w}$. In particular, we have $\mathfrak{H}_v \cap v\mathfrak{H}_w = \mathfrak{H}_{M,v} \cap v\mathfrak{H}_{M,w}$. Lemma 2.26 (i) says $\mathfrak{H}_{vw} = (\mathfrak{H}_v \cup v\mathfrak{H}_w) \smallsetminus (\mathfrak{H}_v \cap v\mathfrak{H}_w)$. Because of $\ell(x) = |\mathfrak{H}_x|$ for all $x \in W$ we obtain

$$\ell(vw) = \ell(v) + \ell(w) - 2 \cdot |\mathfrak{H}_v \cap v\mathfrak{H}_w|.$$

Similarly, we have $\ell_M(vw) = \ell_M(v) + \ell_M(w) - 2 \cdot |\mathfrak{H}_{M,v} \cap v\mathfrak{H}_{M,w}|$. Hence the last claim follows.

(ii) Confer [Abe16a, Lem. 4.1]. From Proposition 1.37 (iii) it follows that $v \leq_M w$ is equivalent to $v^{-1} \leq_M w^{-1}$. Thus, we may assume $w \in W_{M^+}$ or, equivalently, $\mu_{U_P}(w) = 1$. Proposition 4.16 (iii) now shows $\mu_{U_P}(v) = 1$, and hence $v \in W_{M^+}$. □

Before we close this section we want to address one further property of $\mu_{U_P}(w)$. We choose another Levi subgroup L in G with $M \subseteq L$ (see section 1.8; it is constructed from a subset $J \subseteq \Delta$ with $\Delta_M \subseteq J$), and denote by P_L the corresponding parabolic subgroup containing P with unipotent radical U_{P_L}. Then $P \cap L$ is a parabolic subgroup in L with Levi component M and unipotent radical $U_P \cap L$.

Lemma 4.19. *Let $w \in W_M$. Then we have $\mu_{U_P}(w) = \mu_{U_P\cap L}(w) \cdot \mu_{U_{P_L}}(w)$.*

Proof. By our observation in Notation 4.12 we may assume $w \in \Lambda$. As before, we appeal to Step 2 in the proof of Lemma 4.13 to obtain

$$\begin{aligned}
\mu_{U_P}(w) &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_M\\ \langle\alpha,\nu(w)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(w)\rangle)}\right| \\
&= \prod_{\substack{\alpha\in\Sigma_L^+\smallsetminus\Sigma_M\\ \langle\alpha,\nu(w)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(w)\rangle)}\right| \cdot \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_L\\ \langle\alpha,\nu(w)\rangle>0}} \left|U_{(\alpha,0)}/U_{(\alpha,\langle\alpha,\nu(w)\rangle)}\right| \\
&= \mu_{U_P\cap L}(w)\cdot\mu_{U_{P_L}}(w).
\end{aligned}$$

□

4.5. The algebras $\mathcal{H}_R(M,G)$

4.5.1. Definition and two homomorphisms

Recall maps $\Theta^P_{M,R}\colon H_R\big(I_P(1),P\big)\to\mathcal{H}_R(M)$ and $\Xi^P_{G,R}\colon H_R\big(I_P(1),P\big)\to\mathcal{H}_R(G)$ constructed in sections 4.1 and 4.3, respectively. By construction, both maps factor through the R-algebra $R\otimes_{\mathbb{Z}}\operatorname{Im}\Theta^P_{M,\mathbb{Z}}$ (cf. Proposition 4.14 for the construction of $\Xi^P_{G,R}$).

Definition 4.20. We write $\mathcal{H}_R(M,G) := R\otimes_{\mathbb{Z}}\operatorname{Im}\Theta^P_{M,\mathbb{Z}}$. Notice that $\mathcal{H}_R(G,G) = \mathcal{H}_R(G)$, because then $P=G$ and $\Theta^G_{G,\mathbb{Z}} = \mathrm{id}_{\mathcal{H}_{\mathbb{Z}}(G)}$. Given $w\in W_M(1)$, we define

$$\tau^{M,G}_w := 1\otimes\mu_{U_P}(w)T^M_w \in \mathcal{H}_R(M,G).$$

Then $\big(\tau^{M,G}_w\big)_{w\in W_M(1)}$ is an R-basis of $\mathcal{H}_R(M,G)$ by Corollary 4.8. Denote by

$$\begin{aligned}
\theta^{M,G}_M &\colon \mathcal{H}_R(M,G)\longrightarrow\mathcal{H}_R(M), \quad\text{and}\\
\xi^G_{G,M} &\colon \mathcal{H}_R(M,G)\longrightarrow\mathcal{H}_R(G)
\end{aligned}$$

the maps induced by Θ^P_M and Ξ^P_G, respectively.

Remark 4.21. In $\mathcal{H}_R(M,G)$ we have the following braid relation: given $v,w\in W_M(1)$ with $\ell(vw)=\ell(v)+\ell(w)$, we have $\tau^{M,G}_v\cdot\tau^{M,G}_w = \tau^{M,G}_{vw}$.

Proof. We may assume $R=\mathbb{Z}$. The requirement $\ell(vw)=\ell(v)+\ell(w)$ is equivalent to $q_{v,w}=1$. By Proposition 4.16 (ii) we have $\frac{\mu_{U_P}(v)\mu_{U_P}(w)}{\mu_{U_P}(vw)}\cdot q_{M,v,w}=1$. Hence, $\mu_{U_P}(vw) = \mu_{U_P}(v)\cdot\mu_{U_P}(w)$ and $\ell_M(vw)=\ell_M(v)+\ell_M(w)$. The braid relations in $\mathcal{H}_{\mathbb{Z}}(M)$ imply

$$\tau^{M,G}_v\cdot\tau^{M,G}_w = \mu_{U_P}(v)\mu_{U_P}(w)\cdot T^M_vT^M_w = \mu_{U_P}(vw)\cdot T^M_{vw} = \tau^{M,G}_{vw}.$$

□

Given a Levi subgroup $\mathbf{L}$ in $\mathbf{G}$, we will denote by $\mathbf{P}_L$ the corresponding parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}_{P_L}$. For Levi subgroups $\mathbf{M}$ and $\mathbf{L}$ in $\mathbf{G}$ with $\mathbf{M}\subseteq\mathbf{L}$ we will construct R-algebra homomorphisms $\theta^{L,G}_M\colon\mathcal{H}_R(M,G)\to\mathcal{H}_R(M,L)$ and $\xi^G_{L,M}\colon\mathcal{H}_R(M,G)\to\mathcal{H}_R(L,G)$ and show that they behave nicely with respect to composition.

Lemma 4.22. *Let* $\mathbf{M}, \mathbf{L}, \mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$ *with* $\mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{L}'$*. The map*

$$\theta_M^{L,G} : \mathcal{H}_R(M,G) \longrightarrow \mathcal{H}_R(M,L), \quad \tau_w^{M,G} \mapsto \mu_{U_{P_L}}(w) \cdot \tau_w^{M,L} \tag{4.5.1}$$

is a homomorphism of R-algebras. Moreover, the diagram

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\theta_M^{L',G}"] \arrow[dr, "\theta_M^{L,G}"'] & \mathcal{H}_R(M,L') \arrow[d, "\theta_M^{L,L'}"] \\ & \mathcal{H}_R(M,L) \end{tikzcd} \tag{4.5.2}$$

is commutative, i. e. we have $\theta_M^{L,G} = \theta_M^{L,L'} \circ \theta_M^{L',G}$.

Proof. We may assume $R = \mathbb{Z}$, because then the statement follows by extension of scalars. Then $\mathcal{H}_{\mathbb{Z}}(M,G)$ and $\mathcal{H}_{\mathbb{Z}}(M,L)$ are subalgebras of $\mathcal{H}_{\mathbb{Z}}(M)$. Appealing to Lemma 4.19, we compute, inside $\mathcal{H}_{\mathbb{Z}}(M)$,

$$\tau_w^{M,G} = \mu_{U_P}(w) T_w^M = \mu_{U_{P_L}}(w) \cdot \mu_{U_P \cap L}(w) T_w^M = \mu_{U_{P_L}}(w) \tau_w^{M,L}$$

for all $w \in W_M(1)$. Hence, $\theta_M^{L,G}$, being the inclusion map, is a homomorphism of $\mathbb{Z}$-algebras.

Applying Lemma 4.19 again, we obtain

$$\begin{aligned}(\theta_M^{L,L'} \circ \theta_M^{L',G})(\tau_w^{M,G}) &= \theta_M^{L,L'}\big(\theta_M^{L',G}(\tau_w^{M,G})\big) = \theta_M^{L,L'}\big(\mu_{U_{P_{L'}}}(w)\tau_w^{M,L'}\big) \\ &= \mu_{U_{P_{L'}}}(w)\mu_{U_{P_L} \cap L'}(w) \cdot \tau_w^{M,L} = \mu_{U_{P_L}}(w)\tau_w^{M,L} = \theta_M^{L,G}(\tau_w^{M,G})\end{aligned}$$

for all $w \in W_M(1)$. This proves $\theta_M^{L,L'} \circ \theta_M^{L',G} = \theta_M^{L,G}$. □

Lemma 4.23. *Let* $\mathbf{M}, \mathbf{L}, \mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$ *with* $\mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{L}'$*.*

(i) There exists a unique injective R-algebra homomorphism $\xi_{L,M}^G : \mathcal{H}_R(M,G) \to \mathcal{H}_R(L,G)$ which is natural in R and makes the diagram

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\xi_{L,M}^G"] \arrow[d, "\theta_M^{L,G}"'] & \mathcal{H}_R(L,G) \arrow[d, "\theta_L^{L,G}"] \\ \mathcal{H}_R(M,L) \arrow[r, "\xi_{L,M}^L"'] & \mathcal{H}_R(L) \end{tikzcd} \tag{4.5.3}$$

commutative.

(ii) The diagram

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\xi_{L,M}^G"] \arrow[d, "\theta_M^{L',G}"'] & \mathcal{H}_R(L,G) \arrow[d, "\theta_L^{L',G}"] \\ \mathcal{H}_R(M,L') \arrow[r, "\xi_{L,M}^{L'}"'] & \mathcal{H}_R(L,L') \end{tikzcd} \tag{4.5.4}$$

is commutative, i. e. we have $\theta_L^{L',G} \circ \xi_{L,M}^G = \xi_{L,M}^{L'} \circ \theta_M^{L',G}$.

(iii) The diagram

$$\begin{tikzcd} \mathcal{H}_R(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[dr, "\xi^G_{L',M}"'] & \mathcal{H}_R(L,G) \arrow[d, "\xi^G_{L',L}"] \\ & \mathcal{H}_R(L',G) \end{tikzcd} \tag{4.5.5}$$

is commutative, i. e. we have $\xi^G_{L',M} = \xi^G_{L',L} \circ \xi^G_{L,M}$.

Proof. (i) *Step 1:* Assume that p is not a zero-divisor in R. Then $\theta^{L,G}_M$ and $\theta^{L,G}_L$ are inclusions. Hence, the uniqueness of $\xi^G_{L,M}$ is clear provided that it exists. It suffices to prove that $\xi^L_{L,M}$ maps $\mathcal{H}_R(M,G)$ into $\mathcal{H}_R(L,G)$. Let $w \in W_M(1)$. By construction (see the proof of Proposition 4.14) we have $\xi^L_{L,M}(\tau^{M,L}_w) = T^L_w + \sum_{v<_L w} \lambda_v T^L_v$ for some $\lambda_v \in \mathbb{Z}$, where "$<_L$" denotes the Bruhat order in $W_L(1)$. By Lemma 4.19 we have $\mu_{U_P}(w) = \mu_{U_{P_L}}(w)\mu_{U_P\cap L}(w)$, i. e. $\tau^{M,G}_w = \mu_{U_{P_L}}(w)\cdot\tau^{M,L}_w$, and Proposition 4.16 (iii) says that $\mu_{U_{P_L}}(v) \le \mu_{U_{P_L}}(w)$ for all $v \in W_L(1)$ with $v <_L w$. Thus, we have

$$\begin{aligned} \xi^L_{L,M}(\tau^{M,G}_w) &= \mu_{U_{P_L}}(w)\cdot\xi^L_{L,M}(\tau^{M,L}_w) = \mu_{U_{P_L}}(w)T^L_w + \sum_{v<_L w}\lambda_v\mu_{U_{P_L}}(w)T^L_v \\ &= \tau^{L,G}_w + \sum_{v<_L w}\lambda'_v\tau^{L,G}_v \in \mathcal{H}_R(L,G). \end{aligned} \tag{4.5.6}$$

Hence, we obtain a well-defined embedding $\xi^G_{L,M}\colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(L,G)$ making (4.5.3) commutative.

Step 2: The general case. The existence of $\xi^G_{L,M}$ follows from Step 1 by extension of scalars from $\mathbb{Z}$ to R; it is then clear that the commutativity of (4.5.3) holds and that this construction is natural in R. Moreover, it is apparent from (4.5.6) that $\xi^G_{L,M}$ is injective. If p is not a zero-divisor in R this definition does not contradict the one in Step 1, since we have proved uniqueness in this case. Hence, we may assume that p *is* a zero-divisor in R. Consider the ring $S = \mathbb{Z}[X_r \mid r \in R]$ together with the surjective ring homomorphism $\mathrm{pr}\colon S \to R$, $f(X_r)_{r\in R} \mapsto f(r)_{r\in R}$. Notice that p is not a zero-divisor in S so that $\xi^G_{L,M}\colon \mathcal{H}_S(M,G) \to \mathcal{H}_S(L,G)$ is unique. By naturality the diagram

$$\begin{tikzcd} \mathcal{H}_S(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[d, two heads, "\mathrm{pr}\otimes\mathrm{id}"'] & \mathcal{H}_S(L,G) \arrow[d, two heads, "\mathrm{pr}\otimes\mathrm{id}"] \\ \mathcal{H}_R(M,G) \arrow[r, "\xi^G_{L,M}"'] & \mathcal{H}_R(L,G) \end{tikzcd}$$

commutes. Hence, the uniqueness of $\xi^G_{L,M}\colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(L,G)$ follows.

(ii) By naturality, we may assume $R = \mathbb{Z}$. We have a diagram

$$\begin{tikzcd}
\mathcal{H}_{\mathbb{Z}}(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[d, "\theta^{L',G}_M"'] \arrow[dd, bend right, "\theta^{L,G}_M"'] & \mathcal{H}_{\mathbb{Z}}(L,G) \arrow[d, "\theta^{L',G}_L"] \arrow[dd, bend left, "\theta^{L,G}_L"] \\
\mathcal{H}_{\mathbb{Z}}(M,L') \arrow[r, "\xi^{L'}_{L,M}"] \arrow[d, "\theta^{L,L'}_M"'] & \mathcal{H}_{\mathbb{Z}}(L,L') \arrow[d, "\theta^{L,L'}_L"] \\
\mathcal{H}_{\mathbb{Z}}(M,L) \arrow[r, "\xi^L_{L,M}"'] & \mathcal{H}_{\mathbb{Z}}(L)
\end{tikzcd}$$

with the outer and lower square being commutative by (i) (and Lemma 4.22). As $\theta^{L,L'}_L$ is injective, this implies the commutativity of the upper square, i. e. of (4.5.4).

(iii) *Step 1:* Assume $R = \mathbb{Z}[p^{-1}]$. In this case the maps $\theta^{M,G}_M$, $\theta^{L,G}_L$, $\theta^{L',G}_{L'}$, $\theta^{L,G}_M$, $\theta^{L',G}_L$ and $\theta^{L',G}_M$ are the identity morphisms and hence we have $\xi^G_{L,M} = \xi^L_{L,M}$, $\xi^G_{L',L} = \xi^{L'}_{L',L}$ and $\xi^G_{L',M} = \xi^{L'}_{L',M}$ by (ii). Therefore, it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H}_{\mathbb{Z}[p^{-1}]}(M) \arrow[r, "\xi^G_{L,M}"] \arrow[dr, "\xi^G_{L',M}"'] & \mathcal{H}_{\mathbb{Z}[p^{-1}]}(L) \arrow[d, "\xi^G_{L',L}"] \\
& \mathcal{H}_{\mathbb{Z}[p^{-1}]}(L').
\end{tikzcd}$$

Let $w \in W_M(1)$ and take a strictly positive element in M with image $\lambda \in \Lambda(1)$. Let $n \in \mathbb{N}$ such that $e^{n\lambda}w \in W^G_{M^+}(1)$; this is possible by Proposition 1.62. From Proposition 1.62 it follows that $n\lambda$ and $e^{n\lambda}w$ are contained in $W^L_{M^+}(1)$, $W^{L'}_{M^+}(1)$, and in $W^{L'}_{L^+}(1)$. Hence, we have $\xi^G_{L,M}(T^M_w) = T^{L,-1}_{n\lambda} \cdot T^L_{e^{n\lambda}w}$ and $\xi^G_{L',M}(T^M_w) = T^{L',-1}_{n\lambda} \cdot T^{L'}_{e^{n\lambda}w}$, as well as $\xi^G_{L',L}(T^L_{n\lambda}) = T^{L'}_{n\lambda}$ and $\xi^G_{L',L}(T^L_{e^{n\lambda}w}) = T^{L'}_{e^{n\lambda}w}$. Put together we obtain

$$\begin{aligned}
(\xi^G_{L',L} \circ \xi^G_{L,M})(T^M_w) &= \xi^G_{L',L}\big(\xi^G_{L,M}(T^M_w)\big) = \xi^G_{L',L}\big(T^{L,-1}_{n\lambda} \cdot T^L_{e^{n\lambda}w}\big) \\
&= \big(\xi^G_{L',L}(T^L_{n\lambda})\big)^{-1} \cdot \xi^G_{L',L}(T^L_{e^{n\lambda}w}) \\
&= T^{L',-1}_{n\lambda} \cdot T^{L'}_{e^{n\lambda}w} = \xi^G_{L',M}(T^M_w).
\end{aligned}$$

Step 2: Assume $R = \mathbb{Z}$. We have inclusions $\mathcal{H}_{\mathbb{Z}}(M,G) \subseteq \mathcal{H}_{\mathbb{Z}[p^{-1}]}(M,G)$, $\mathcal{H}_{\mathbb{Z}}(L,G) \subseteq \mathcal{H}_{\mathbb{Z}[p^{-1}]}(L,G)$ and $\mathcal{H}_{\mathbb{Z}}(L',G) \subseteq \mathcal{H}_{\mathbb{Z}[p^{-1}]}(L',G)$. Now, the claim follows from Step 1 by restriction (and naturality).

Step 3: The general case. This is clear by extension of scalars from $\mathbb{Z}$ to R. □

4.5.2. Redefinition of parabolic induction

Definition 4.24. Let $\mathcal{M}$ be a right $\mathcal{H}_R(M)$-module. View $\mathcal{M}$ via the homomorphism $\theta^{M,G}_M \colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(M)$ as a right $\mathcal{H}_R(M,G)$-module. We use the morphism

$\xi^G_{G,M}\colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(G)$ to define a right $\mathcal{H}_R(G)$-module

$$\mathcal{M} \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(G).$$

In this way we obtain a functor $-\otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(G)$ from the category of right $\mathcal{H}_R(M)$-modules to the category of right $\mathcal{H}_R(G)$-modules.

We recall the R-algebra morphism $\theta^+\colon \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ (Proposition 2.28). It is used to define the parabolic induction functor $-\otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G)$ from the category of right $\mathcal{H}_R(M)$-modules to the category of right $\mathcal{H}_R(G)$-modules (see [Vig15], [OV18]).

Theorem 4.25. *Let $\mathcal{M}$ be a right $\mathcal{H}_R(M)$-module. Then $\mathcal{M} \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(G)$ coincides with $\mathcal{M} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G)$ as a right $\mathcal{H}_R(G)$-module.*

Proof. Let $\mathcal{N}$ be an arbitrary R-module and let $\rho\colon \mathcal{M} \times \mathcal{H}_R(G) \to \mathcal{N}$ be an R-bilinear map. We claim that the following conditions are equivalent:

(a) $\rho(m \cdot f, h) = \rho(m, \theta^+(f) \cdot h)$ for all $m \in \mathcal{M}$, $f \in \mathcal{H}_R(M^+)$, and $h \in \mathcal{H}_R(G)$.

(b) $\rho(m \cdot \theta^{M,G}_M(f), h) = \rho(m, \xi^G_{G,M}(f) \cdot h)$ for all $m \in \mathcal{M}$, $f \in \mathcal{H}_R(M,G)$, and $h \in \mathcal{H}_R(G)$.

Assume (b). Let $w \in W_{M^+}(1)$. Then $\theta^{M,G}_M(\tau^{M,G}_w) = T^M_w$ and $\xi^G_{G,M}(\tau^{M,G}_w) = T_w = \theta^+(T^M_w)$ by construction, whence (a).

Assume (a). Let $w \in W_M(1)$. Take any strictly positive $\lambda \in \Lambda(1)$, i. e. an element lifted by a central element in M with $\langle \alpha, \nu(\lambda)\rangle = 0$ for all $\alpha \in \Sigma_M$ and $\langle \alpha, \nu(\lambda)\rangle < 0$ for all $\alpha \in \Sigma^+ \setminus \Sigma_M$ (it exists by Remark 1.61, see also Proposition 1.62). Let $n \in \mathbb{N}$ with $e^{n\lambda}w \in W_{M^+}(1)$. Since $n\lambda \in \Omega_M$ we have $\tau^{M,G}_{n\lambda} \cdot \tau^{M,G}_w = \mu_{U_P}(w) \cdot \tau^{M,G}_{e^{n\lambda}w}$. By construction we have $\theta^{M,G}_M(\tau^{M,G}_{n\lambda}) = T^M_{n\lambda}$, $\xi^G_{G,M}(\tau^{M,G}_{e^{n\lambda}w}) = \theta^+(T^M_{e^{n\lambda}w})$, and $\xi^G_{G,M}(\tau^{M,G}_{n\lambda}) = T_{n\lambda} = \theta^+(T^M_{n\lambda})$. As $\theta^{M,G}_M$ and $\xi^G_{G,M}$ are R-algebra homomorphisms, we compute

$$\begin{aligned}
\rho(m \cdot \theta^{M,G}_M(\tau^{M,G}_w), h) &= \rho(m \cdot T^{M,-1}_{n\lambda} \cdot \theta^{M,G}_M(\tau^{M,G}_{n\lambda}\tau^{M,G}_w), h) \\
&= \rho(m \cdot T^{M,-1}_{n\lambda} \cdot \mu_{U_P}(w) \cdot \theta^{M,G}_M(\tau^{M,G}_{e^{n\lambda}w}), h) \\
&= \rho(m \cdot T^{M,-1}_{n\lambda}, \mu_{U_P}(w) \cdot \xi^G_{G,M}(\tau^{M,G}_{e^{n\lambda}w}) \cdot h) \\
&= \rho(m \cdot T^{M,-1}_{n\lambda}, \xi^G_{G,M}(\tau^{M,G}_{n\lambda} \cdot \tau^{M,G}_w) \cdot h) \\
&= \rho(m \cdot T^{M,-1}_{n\lambda}, \theta^+(T^M_{n\lambda}) \cdot \xi^G_{G,M}(\tau^{M,G}_w) \cdot h) \\
&= \rho(m \cdot T^{M,-1}_{n\lambda} T^M_{n\lambda}, \xi^G_{G,M}(\tau^{M,G}_w) \cdot h) \\
&= \rho(m, \xi^G_{G,M}(\tau^{M,G}_w) \cdot h).
\end{aligned}$$

Thus, (b) holds.

The discussion above shows that $\mathcal{M} \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(G)$ and $\mathcal{M} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G)$ satisfy the same universal property, and hence coincide as R-modules. As the right $\mathcal{H}_R(G)$-module structure is given by right multiplication for both modules, the theorem follows. □

4.5.3. The transitivity property of tensor products

Proposition 4.26. *Let* $\mathbf{M}, \mathbf{L}$ *be Levi subgroups in* $\mathbf{G}$ *with* $\mathbf{M} \subseteq \mathbf{L}$. *Let* $\lambda \in \Lambda(1)$ *be a strictly* L*-positive element, i. e. lifted by a central element in* M *and satisfying* $\langle \alpha, \nu(\lambda) \rangle = 0$ *for all* $\alpha \in \Sigma_L$ *and* $\langle \alpha, \nu(\lambda)\rangle < 0$ *for all* $\alpha \in \Sigma^+ \setminus \Sigma_L$. *Then* $\mathcal{H}_R(M,L)$ *is the localization of* $\mathcal{H}_R(M,G)$ *at the central element* $\tau_\lambda^{M,G}$.

Proof. We will show that $\theta_M^{L,G} \colon \mathcal{H}_R(M,G) \to \mathcal{H}_R(M,L)$ induces an R-algebra isomorphism $\mathcal{H}_R(M,G)[(\tau_\lambda^{M,G})^{-1}] \cong \mathcal{H}_R(M,L)$. It is important to notice that the element $\theta_M^{L,G}(\tau_{n\lambda}^{M,G}) = \mu_{U_{P_L}}(n\lambda) \cdot \tau_{n\lambda}^{M,L} = \tau_{n\lambda}^{M,L}$ is central and invertible with inverse $\tau_{-n\lambda}^{M,L}$ in $\mathcal{H}_R(M,L)$ for each $n \in \mathbb{N}$. We obtain a well-defined R-algebra homomorphism

$$\widetilde{\theta}_M^{L,G} \colon \mathcal{H}_R(M,G)\big[(\tau_\lambda^{M,G})^{-1}\big] \longrightarrow \mathcal{H}_R(M,L), \quad \frac{\tau_w^{M,G}}{\tau_{n\lambda}^{M,G}} \longmapsto \tau_{-n\lambda}^{M,L} \cdot \theta_M^{L,G}(\tau_w^{M,G}).$$

We describe the inverse map. Let $w \in W_M(1)$. There exists $n \in \mathbb{N}$ such that $e^{n\lambda}w \in W_{L^+}(1)$, and hence $\mu_{U_{P_L}}(e^{n\lambda}w) = 1$. As $n\lambda \in \Omega_L$ we have $\ell_L(e^{n\lambda}w) = \ell_L(n\lambda) + \ell_L(w)$. Remark 4.21 now implies

$$\tau_w^{M,L} = \tau_{-n\lambda}^{M,L} \cdot \tau_{n\lambda}^{M,L} \cdot \tau_w^{M,L} = \tau_{-n\lambda}^{M,L} \cdot \tau_{e^{n\lambda}w}^{M,L} = \tau_{-n\lambda}^{M,L} \cdot \theta_M^{L,G}\big(\tau_{e^{n\lambda}w}^{M,G}\big). \tag{4.5.7}$$

Let $m \in \mathbb{N}$ be another integer with $e^{m\lambda}w \in W_{L^+}(1)$. We may assume $m \geq n$. As $(m-n)\lambda \in \Omega_L$, we have $\ell_L(e^{m\lambda}w) = \ell_L\big((m-n)\lambda\big) + \ell_L(e^{n\lambda}w)$. But then we also have $\ell(e^{m\lambda}w) = \ell\big((m-n)\lambda\big) + \ell(e^{n\lambda}w)$ by Corollary 4.18, using the fact that $(m-n)\lambda, e^{n\lambda}w \in W_{L^+}(1)$. Remark 4.21 again implies $\tau_{(m-n)\lambda}^{M,G} \cdot \tau_{e^{n\lambda}w}^{M,G} = \tau_{e^{m\lambda}w}^{M,G}$. The discussion shows that the map given by

$$\gamma \colon \mathcal{H}_R(M,L) \longrightarrow \mathcal{H}_R(M,G)\big[(\tau_\lambda^{M,G})^{-1}\big], \quad \tau_w^{M,L} \longmapsto \frac{\tau_{e^{n\lambda}w}^{M,G}}{\tau_{n\lambda}^{M,G}}$$

(and R-linear extension) is independent of the choice of n. By (4.5.7) we have $\widetilde{\theta}_M^{L,G} \circ \gamma = \mathrm{id}_{\mathcal{H}_R(M,L)}$. Conversely, let $w \in W_M(1)$ and $n \in \mathbb{N}$. Take $m \in \mathbb{N}$ with $e^{m\lambda}w \in W_{L^+}(1)$. Notice that $\mu_{U_{P_L}}(e^{m\lambda}w) = 1$. As $m\lambda$ is lifted by a central element in L we have $\mu_{U_P \cap L}(e^{m\lambda}w) = \mu_{U_P\cap L}(w)$. Using Lemma 4.19 we compute

$$\begin{aligned}\mu_{U_{P_L}}(w) \cdot \mu_{U_P}(e^{m\lambda}w) &= \mu_{U_{P_L}}(w) \cdot \mu_{U_P\cap L}(e^{m\lambda}w) \cdot \mu_{U_{P_L}}(e^{m\lambda}w)\\ &= \mu_{U_{P_L}}(w) \cdot \mu_{U_P \cap L}(w) = \mu_{U_P}(w).\end{aligned}$$

From this calculation we infer $\tau^{M,G}_{m\lambda}\cdot\tau^{M,G}_{w} = \mu_{U_{P_L}}(w)\cdot\tau^{M,G}_{e^{m\lambda}w}$. Thus, we compute

$$\begin{aligned}
\big(\gamma\circ\widetilde{\theta}^{L,G}_{M}\big)\left(\frac{\tau^{M,G}_{w}}{\tau^{M,G}_{n\lambda}}\right) &= \gamma\big(\tau^{M,L}_{-n\lambda}\cdot\theta^{L,G}_{M}(\tau^{M,G}_{w})\big) = \gamma\big(\mu_{U_{P_L}}(w)\cdot\tau^{M,L}_{-n\lambda}\cdot\tau^{M,L}_{w}\big)\\
&= \mu_{U_{P_L}}(w)\cdot\gamma\big(\tau^{M,L}_{e^{-n\lambda}w}\big) = \frac{\mu_{U_{P_L}}(w)\cdot\tau^{M,G}_{e^{m\lambda}w}}{\tau^{M,G}_{(n+m)\lambda}}\\
&= \frac{\tau^{M,G}_{m\lambda}\cdot\tau^{M,G}_{w}}{\tau^{M,G}_{m\lambda}\cdot\tau^{M,G}_{n\lambda}} = \frac{\tau^{M,G}_{w}}{\tau^{M,G}_{n\lambda}}.
\end{aligned}$$

This shows $\gamma\circ\widetilde{\theta}^{L,G}_{M} = \mathrm{id}_{\mathcal{H}_R(M,G)[(\tau^{M,G}_{\lambda})^{-1}]}$. Hence, $\widetilde{\theta}^{L,G}_{M}$ is an isomorphism of R-algebras. □

Proposition 4.27. *Let* $\mathbf{M},\mathbf{L},\mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$ *with* $\mathbf{M}\subseteq\mathbf{L}\subseteq\mathbf{L}'$. *The map*

$$\begin{aligned}
\mathcal{H}_R(M,L')\otimes_{\mathcal{H}_R(M,G)}\mathcal{H}_R(L,G) &\longrightarrow \mathcal{H}_R(L,L'),\\
x\otimes y &\longmapsto \xi^{L'}_{L,M}(x)\cdot\theta^{L',G}_{L}(y)
\end{aligned}\tag{4.5.8}$$

is an isomorphism of $\mathcal{H}_R(M,L')$*-*$\mathcal{H}_R(L,G)$*-bimodules.*

Proof. Given $x\in\mathcal{H}_R(M,L')$, $z\in\mathcal{H}_R(M,G)$, and $y\in\mathcal{H}_R(L,G)$, we compute

$$\begin{aligned}
\xi^{L'}_{L,M}\big(x\cdot\theta^{L',G}_{M}(z)\big)\cdot\theta^{L',G}_{L}(y) &= \xi^{L'}_{L,M}(x)\cdot\big(\xi^{L'}_{L,M}\circ\theta^{L',G}_{M}\big)(z)\cdot\theta^{L',G}_{L}(y)\\
&= \xi^{L'}_{L,M}(x)\cdot\big(\theta^{L',G}_{L}\circ\xi^{G}_{L,M}\big)(z)\cdot\theta^{L',G}_{L}(y)\\
&= \xi^{L'}_{L,M}(x)\cdot\theta^{L',G}_{L}\big(\xi^{G}_{L,M}(z)\cdot y\big),
\end{aligned}$$

where we have used Lemma 4.23 (ii) for the second equality. Hence, the map (4.5.8) is well-defined. By construction it preserves the bimodule structure. Let $\lambda\in\Lambda(1)$ be a strictly L'-positive element. By Proposition 4.26 we have isomorphisms two $\mathcal{H}_R(M,L')\cong\mathcal{H}_R(M,G)[(\tau^{M,G}_{\lambda})^{-1}]$ and $\mathcal{H}_R(L,L')\cong\mathcal{H}_R(L,G)[(\tau^{L,G}_{\lambda})^{-1}]$. Under these identifications the map (4.5.8) reads

$$\begin{aligned}
\mathcal{H}_R(M,G)\big[(\tau^{M,G}_{\lambda})^{-1}\big]\otimes_{\mathcal{H}_R(M,G)}\mathcal{H}_R(L,G) &\longrightarrow \mathcal{H}_R(L,G)\big[(\tau^{L,G}_{\lambda})^{-1}\big],\\
\frac{1}{\tau^{M,G}_{n\lambda}}\otimes\tau^{L,G}_{w} &\longmapsto \frac{\tau^{L,G}_{w}}{\tau^{L,G}_{n\lambda}}
\end{aligned}$$

which is clearly an R-linear isomorphism (notice that $\xi^{G}_{L,M}(\tau^{M,G}_{n\lambda}) = \tau^{L,G}_{n\lambda}$). □

Theorem 4.28. *Let* $\mathbf{M},\mathbf{M}',\mathbf{M}'',\mathbf{L},$ *and* $\mathbf{L}'$ *be Levi subgroups in* $\mathbf{G}$ *arranged as follows:*

$$\begin{array}{ccccc}
\mathbf{L} & \subseteq & \mathbf{L}' & \subseteq & \mathbf{G}\\
\cup\!| & & \cup\!| & & \cup\!|\\
\mathbf{M} & \subseteq & \mathbf{M}' & \subseteq & \mathbf{M}''.
\end{array}$$

Then the map

$$\mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M'',G) \longrightarrow \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M',L') \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G)$$
$$x \otimes y \longmapsto x \otimes 1 \otimes y$$

is an isomorphism of $\mathcal{H}_R(M,L)$-$\mathcal{H}_R(M'',G)$-bimodules.

Proof. Using Lemmas 4.22 and 4.23 and Proposition 4.27 we have R-linear isomorphisms

$$\begin{aligned}
&\mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M'',G) \\
&\qquad \cong \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M,L') \underset{\mathcal{H}_R(M,G)}{\otimes} \mathcal{H}_R(M',G) \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G) \\
&\qquad \cong \mathcal{H}_R(M,L) \underset{\mathcal{H}_R(M,L')}{\otimes} \mathcal{H}_R(M',L') \underset{\mathcal{H}_R(M',G)}{\otimes} \mathcal{H}_R(M'',G).
\end{aligned}$$

The composite of these isomorphisms sends $x \otimes y$ to $x \otimes 1 \otimes y$ which preserves the bimodule structure. The theorem is proved. □

As a corollary we deduce the transitivity of the parabolic induction functor:

Corollary 4.29. *Let* $\mathbf{M}, \mathbf{L}$ *be Levi subgroups in* $\mathbf{G}$ *with* $\mathbf{M} \subseteq \mathbf{L}$. *Let* $\mathcal{M}$ *be a right* $\mathcal{H}_R(M)$-*module. Then we have a natural isomorphism of right* $\mathcal{H}_R(G)$-*modules*

$$\mathcal{M} \otimes_{\mathcal{H}_R(M^{+,L})} \mathcal{H}_R(L) \otimes_{\mathcal{H}_R(L^+)} \mathcal{H}_R(G) \cong \mathcal{M} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G),$$

where $\mathcal{H}_R(M^{+,L})$ *denotes the subalgebra of* $\mathcal{H}_R(M)$ *with basis* $(T_w^M)_{W^L_{M^+}(1)}$.

Proof. Confer [Vig15, Prop. 4.3]. We have a natural $\mathcal{H}_R(G)$-linear isomorphism $\mathcal{M} \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G) \cong \mathcal{M} \otimes_{\mathcal{H}_R(M)} \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G)$ and similarly for $\mathcal{H}_R(G)$ replaced by $\mathcal{H}_R(L)$. By Theorems 4.25 and 4.28 we have $\mathcal{H}_R(G)$-linear isomorphisms

$$\begin{aligned}
\mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G) &\cong \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M,G)} \mathcal{H}_R(G) \\
&\cong \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M,L)} \mathcal{H}_R(L) \otimes_{\mathcal{H}_R(L,G)} \mathcal{H}_R(G) \\
&\cong \mathcal{H}_R(M) \otimes_{\mathcal{H}_R(M^{+,L})} \mathcal{H}_R(L) \otimes_{\mathcal{H}_R(L^+)} \mathcal{H}_R(G).
\end{aligned}$$

The statement follows. □

4.5.4. Alcove walk bases and a filtration

Let $\mathbf{M}$ be a Levi subgroup in $\mathbf{G}$. We will describe an ascending exhaustive filtration on $\mathcal{H}_R(M,G)$. But first, we will define alcove walk bases for $\mathcal{H}_R(M,G)$; these were already indicated in Remark 4.17 (b).

Definition 4.30. Let o be an orientation of $(\mathscr{A}_M, \mathfrak{H}_M)$. Given $w \in W_M(1)$, we define $E_o^{M,G}(w) := \mu_{U_P}(w) \cdot E_o(w)$ in $\mathcal{H}_{\mathbb{Z}}(M)$. By (2.2.7) and Proposition 4.16 (iii) we have

$$E_o^{M,G}(w) = \tau_w^{M,G} + \sum_{v <_M w} \lambda_v \cdot \tau_v^{M,G} \in \mathcal{H}_{\mathbb{Z}}(M,G), \quad \text{for certain } \lambda_v \in \mathbb{Z}. \tag{4.5.9}$$

Clearly, $\big(E_o^{M,G}(w)\big)_{w\in W_M(1)}$ is a $\mathbb{Z}$-basis of $\mathcal{H}_{\mathbb{Z}}(M,G)$. Extending scalars from $\mathbb{Z}$ to R we obtain an R-basis $\big(E_o^{M,G}(w)\big)_{w\in W_M(1)}$ of $\mathcal{H}_R(M,G)$. It is called the *alcove walk basis* of $\mathcal{H}_R(M,G)$ associated with o. By Theorem 2.21 and Proposition 4.16 (ii) we have

$$E_o^{M,G}(w)\cdot E_{o\bullet w}^{M,G}(v) = \frac{\mu_{U_P}(w)\mu_{U_P}(v)}{\mu_{U_P}(wv)}\cdot q_{M,v,w}\cdot E_o^{M,G}(wv) = q_{w,v}\cdot E_o^{M,G}(wv) \quad (4.5.10)$$

for all $w,v\in W_M(1)$. If o is the orientation of $(\mathcal{A}_M,\mathfrak{H}_M)$ with $\mathfrak{C}_M\subseteq H_{o,+}$ for all $H\in\mathfrak{H}_M$, we obtain a new basis $\big(\tau_w^{M,G,*}\big)_{w\in W_M(1)}$ of $\mathcal{H}_R(M,G)$, where

$$\tau_w^{M,G,*} := E_o^{M,G}(w) = 1\otimes\mu_{U_P}(w)\cdot T_w^{M,*}.$$

By Proposition 4.16 (i) and Proposition 2.17 (ii) we have

$$\tau_w^{M,G}\cdot\tau_{w^{-1}}^{M,G,*} = \tau_{w^{-1}}^{M,G,*}\cdot\tau_w^{M,G} = q_w. \quad (4.5.11)$$

Proposition 4.31. *Consider the free R-submodule $\mathcal{F}_n^{M,G} := \mathcal{F}_n\big(\mathcal{H}_R(M,G)\big)$ of the R-algebra $\mathcal{H}_R(M,G)$ generated by $\big(\tau_w^{M,G}\big)_{w\in W_M(1),\mu_{U_P}(w)\le q^n}$, for $n\in\mathbb{Z}_{\ge 0}$. This defines an ascending exhaustive filtration on $\mathcal{H}_R(M,G)$, i. e. we have*

- $1\in\mathcal{F}_0^{M,G}$ *and* $\mathcal{F}_n^{M,G}\cdot\mathcal{F}_m^{M,G}\subseteq\mathcal{F}_{n+m}^{M,G}$ *for all* $n,m\in\mathbb{Z}_{\ge 0}$;
- $\mathcal{F}_n^{M,G}\subseteq\mathcal{F}_{n+1}^{M,G}$ *for all* $n\in\mathbb{Z}_{\ge 0}$;
- $\mathcal{H}_R(M,G) = \bigcup_{n=0}^{\infty}\mathcal{F}_n^{M,G}$.

Moreover, $\mathcal{F}_0^{M,G}$ is isomorphic to $\mathcal{H}_R(M^+)$.

Proof. We have $1\in\mathcal{F}_0^{M,G}$, $\mathcal{F}_n^{M,G}\subseteq\mathcal{F}_{n+1}^{M,G}$ for all $n\in\mathbb{Z}_{\ge 0}$, and $\mathcal{H}_R(M,G) = \bigcup_{n=0}^{\infty}\mathcal{F}_n^{M,G}$. In order to show that $(\mathcal{F}_n^{M,G})_{n\ge 0}$ defines an ascending exhaustive filtration of $\mathcal{H}_R(M,G)$, it remains to prove $\mathcal{F}_n^{M,G}\cdot\mathcal{F}_m^{M,G}\subseteq\mathcal{F}_{n+m}^{M,G}$ for all $n,m\in\mathbb{Z}_{\ge 0}$.

Given $v,w\in W_M(1)$ with $v\le_M w$, we have $\mu_{U_P}(v)\le\mu_{U_P}(w)$ by Proposition 4.16 (iii) and hence (4.5.9) implies $E_o^{M,G}(w)\in\mathcal{F}_n^{M,G}$ for all $w\in W_M(1)$ with $\mu_{U_P}(w)\le q^n$. Hence, $\big(E_o^{M,G}(w)\big)_{\mu_{U_P}(w)\le q^n}$ is an R-basis of $\mathcal{F}_n^{M,G}$ for any orientation o of $(\mathcal{A}_M,\mathfrak{H}_M)$.

Let $w,v\in W_M(1)$ with $\mu_{U_P}(w)\le q^n$ and $\mu_{U_P}(v)\le q^m$. Then Proposition 4.16 (ii) implies $\mu_{U_P}(wv)\le\mu_{U_P}(w)\cdot\mu_{U_P}(v)\le q^{n+m}$. Let o be an orientation of $(\mathcal{A}_M,\mathfrak{H}_M)$. By (4.5.10) we have

$$E_o^{M,G}(w)\cdot E_{o\bullet w}^{M,G}(v) = q_{w,v}\cdot E_o^{M,G}(wv)\in\mathcal{F}_{n+m}^{M,G}.$$

By the remark above this is enough to conclude $\mathcal{F}_n^{M,G}\cdot\mathcal{F}_m^{M,G}\subseteq\mathcal{F}_{n+m}^{M,G}$. It is clear that $\mathcal{F}_0^{M,G}$ identifies with $\mathcal{H}_R(M^+)$ under $\theta_M^{M,G}:\mathcal{H}_R(M,G)\to\mathcal{H}_R(M)$. □

4.6. The algebras $\overline{\mathcal{H}}_R(M,G)$

In the previous section we constructed, given Levi subgroups $\mathbf{M},\mathbf{L}$ inside $\mathbf{G}$ with $\mathbf{M}\subseteq\mathbf{L}$, certain R-algebras $\mathcal{H}_R(M,G)$ together with R-algebra homomorphisms

$\theta_M^{L,G}\colon \mathcal{H}_R(M,G)\to\mathcal{H}_R(M,L)$ and $\xi_{L,M}^G\colon \mathcal{H}_R(M,G)\to\mathcal{H}_R(L,G)$ which behave well under composition. The algebra $\mathcal{H}_R(M,G)$ may be thought of as a replacement for $\mathcal{H}_R(M^+)$. Then $\theta_M^{M,G}$ corresponds to the inclusion $\mathcal{H}_R(M^+)\subseteq\mathcal{H}_R(M)$ and $\xi_{G,M}^G$ corresponds to the embedding $\theta^+\colon \mathcal{H}_R(M^+)\to\mathcal{H}_R(G)$ (2.3.1).

In this section we will define R-algebras $\overline{\mathcal{H}}_R(M,G)$ together with R-algebra homomorphisms $\overline{\theta}_M^{L,G}\colon \overline{\mathcal{H}}_R(M,G)\to\overline{\mathcal{H}}_R(M,L)$ and $\overline{\xi}_{L,M}^{*,G}\colon \overline{\mathcal{H}}_R(M,G)\to\overline{\mathcal{H}}_R(L,G)$. Then $\overline{\mathcal{H}}_R(M,G)$ may be thought of as a replacement for $\mathcal{H}_R(M^-)$, where $\overline{\theta}_M^{M,G}$ corresponds to the inclusion $\mathcal{H}_R(M^-)\subseteq\mathcal{H}_R(M)$, and $\overline{\xi}_{G,M}^{*,G}$ corresponds to the embedding $\theta^{*,-}\colon \mathcal{H}_R(M^-)\to\mathcal{H}_R(G)$ given by $T_w^{M,*}\mapsto T_w^*$.

Recall the R-algebra homomorphism $\Theta_{M,R}^P\colon H_R(I_P(1),P)\to\mathcal{H}_R(M)$. By Proposition 2.10 we have anti-automorphisms $\zeta_P\colon H_R(I_P(1),P)\to H_R(I_P(1),P)$ given by $(g)_{I_P(1)}\mapsto(g^{-1})_{I_P(1)}$ and $\zeta_M\colon\mathcal{H}_R(M)\to\mathcal{H}_R(M)$ given by $T_m^M\mapsto T_{m^{-1}}^M$. Then $\overline{\Theta}_M^P:=\overline{\Theta}_{M,R}^P:=\zeta_M\circ\Theta_{M,R}^P\circ\zeta_P$ defines another R-algebra homomorphism

$$\overline{\Theta}_{M,R}^P\colon H_R(I_P(1),P)\longrightarrow\mathcal{H}_R(M),\quad (g)_{I_P(1)}\longmapsto \nu_M(g^{-1})\mu_{U_P}(g^{-1})\cdot T_{g_M}^M.$$

By Proposition 4.7 the set $\{\mu_{U_P}(m^{-1})T_m^M \mid m\in M\}$ is a $\mathbb{Z}$-basis of $\operatorname{Im}\overline{\Theta}_{M,\mathbb{Z}}^P$.

Definition 4.32. Let $\mathbf{M}$ be a Levi subgroup in $\mathbf{G}$.

(a) We write $\overline{\mathcal{H}}_R(M,G):=R\otimes_{\mathbb{Z}}\operatorname{Im}\overline{\Theta}_{M,\mathbb{Z}}^P$. It is an R-algebra and free as an R-module with basis $(\overline{\tau}_w^{M,G})_{w\in W_M(1)}$, where

$$\overline{\tau}_w^{M,G}:=1\otimes\mu_{U_P}(w^{-1})T_w^M,\quad \text{for } w\in W_M(1).$$

(b) Let o be an orientation of $(\mathcal{A}_M,\mathfrak{H}_M)$. As for $\mathcal{H}_R(M,G)$ we define an element $\overline{E}_o^{M,G}(w):=1\otimes\mu_{U_P}(w^{-1})E_o(w)$ of $\overline{\mathcal{H}}_R(M,G)=R\otimes_{\mathbb{Z}}\overline{\mathcal{H}}_{\mathbb{Z}}(M,G)$ for each $w\in W_M(1)$. Because of Proposition 4.16, (iii) we have

$$\overline{E}_o^{M,G}(w)=\overline{\tau}_w^{M,G}+\sum_{v<_M w}\lambda_v\overline{\tau}_v^{M,G},\quad \text{for certain } \lambda_v\in\mathbb{Z}.$$

Hence, $(\overline{E}_o^{M,G}(w))_{w\in W_M(1)}$ is an R-basis of $\overline{\mathcal{H}}_R(M,G)$.

In particular, if o orients $(\mathcal{A}_M,\mathfrak{H}_M)$ towards the fundamental alcove $\mathfrak{C}_M$, i. e. such that $\mathfrak{C}_M\subseteq H_{o,+}$ for all $H\in\mathfrak{H}_M$, then we have $E_o(w)=T_w^{M,*}$ for all $w\in W_M(1)$. Hence, we obtain an R-basis $(\overline{\tau}_w^{M,G,*})_{w\in W_M(1)}$ of $\overline{\mathcal{H}}_R(M,G)$, where

$$\overline{\tau}_w^{M,G,*}:=\overline{E}_o^{M,G}(w)=1\otimes\mu_{U_P}(w^{-1})T_w^{M,*}\quad \text{for } w\in W_M(1).$$

Let $\mathbf{M},\mathbf{L}$ be Levi subgroups in $\mathbf{G}$ with $\mathbf{M}\subseteq\mathbf{L}$. We now come to the definition of the maps $\overline{\theta}_M^{L,G}\colon\overline{\mathcal{H}}_R(M,G)\to\overline{\mathcal{H}}_R(M,L)$ and $\overline{\xi}_{L,M}^{G,*}\colon\overline{\mathcal{H}}_R(M,G)\to\overline{\mathcal{H}}_R(L,G)$.

First, we define the R-linear map

$$\overline{\theta}_M^{L,G}\colon\overline{\mathcal{H}}_R(M,G)\longrightarrow\overline{\mathcal{H}}_R(M,L),\quad \overline{\tau}_w^{M,G,*}\longmapsto\mu_{U_{P_L}}(w^{-1})\cdot\overline{\tau}_w^{M,L,*}.$$

Lemma 4.33. *Let* **M**, **L**, *and* **L**$'$ *be Levi subgroups in* **G** *with* **M** $\subseteq$ **L** $\subseteq$ **L**$'$. *The map* $\overline{\theta}_M^{L,G}$ *is a homomorphism of* R*-algebras. Moreover, we have a commutative diagram*

$$\begin{tikzcd} \overline{\mathcal{H}}_R(M,G) \arrow[r, "\overline{\theta}_M^{L',G}"] \arrow[dr, "\overline{\theta}_M^{L,G}"'] & \overline{\mathcal{H}}_R(M,L') \arrow[d, "\overline{\theta}_M^{L,L'}"] \\ & \overline{\mathcal{H}}_R(M,L). \end{tikzcd}$$

Proof. After reducing to $R = \mathbb{Z}$ we see that $\overline{\theta}_M^{L,G}$ is just an inclusion and in particular multiplicative. The commutativity of the diagram is another application of Lemma 4.19. Notice that this also shows

$$\overline{\theta}_M^{L,G}\left(\overline{\tau}_w^{M,G}\right) = \mu_{U_{P_L}}(w^{-1}) \cdot \overline{\tau}_w^{M,L}, \quad \text{for all } w \in W_M(1),$$

which justifies the notation $\overline{\theta}_M^{L,G}$ over $\overline{\theta}_M^{L,G,*}$. □

In order to define $\overline{\xi}_{L,M}^{G,*}$ we first assume $R = \mathbb{Z}$. Let o be the orientation of $(\mathscr{A}_L, \mathfrak{H}_L)$ with $\mathfrak{C}_L \subseteq H_{o,+}$ for all $H \in \mathfrak{H}_L$. Then we have $E_o(w) = T_w^{L,*}$ in $\mathcal{H}_\mathbb{Z}(L)$ for all $w \in W_L(1)$. We fix a strictly negative element $a \in Z(M)$ (relative to L) and denote its image in $\Lambda(1)$ by λ. Let $w \in W_M(1)$. There exists $n \in \mathbb{N}$ such that $e^{n\lambda}w \in W_{M^-}^L(1)$. By Theorem 2.21 we have $E_o(n\lambda) \cdot E_{o\bullet n\lambda}(w) = q_{L,n\lambda,w} \cdot E_o(e^{n\lambda}w)$. Since $n\lambda$ and w commute, it follows directly from the definition of $q_{L,n\lambda,w} = (q_{L,n\lambda}q_{L,w}/q_{L,e^{n\lambda}w})^{1/2}$ and Lemma 4.13 that $q_{L,n\lambda,w} = q_{L,w,n\lambda} = q_{L,-n\lambda,w^{-1}} = \mu_{U_P\cap L}(w^{-1})$. Hence, computing inside $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(L)$, we have

$$\begin{aligned} \overline{\xi}_{L,M}^{G,*}\left(\overline{\tau}_w^{M,G,*}\right) &:= \mu_{U_P}(w^{-1}) \cdot (T_{n\lambda}^{L,*})^{-1} T_{e^{n\lambda}w}^{L,*} \\ &= \mu_{U_{P_L}}(w^{-1}) \cdot E_{o\bullet n\lambda}(w) = \overline{E}_{o\bullet n\lambda}^{L,G}(w) \in \mathcal{H}_\mathbb{Z}(L,G). \end{aligned}$$

This definition does not depend on n as can be seen by copying the argument in the proof of Proposition 2.29 and using $T_v^{L,*} = q_{L,v} \cdot (T_{v^{-1}}^L)^{-1}$ for all $v \in W_L(1)$. More precisely, let $m \in N_M$ be an element lifting $w^{-1} \in W_M(1)$; then ma^{-n} lifts $w^{-1}e^{-n\lambda}$. Inside $\mathcal{H}_{\mathbb{Z}[p^{-1}]}(L)$ we compute

$$\begin{aligned} (T_{n\lambda}^{L,*})^{-1} \cdot T_{e^{n\lambda}w}^{L,*} &= \frac{q_{L,e^{n\lambda}w}}{q_{L,n\lambda}} \cdot T_{-n\lambda}^L \cdot (T_{w^{-1}e^{-n\lambda}}^L)^{-1} \\ &= \frac{q_{L,w}}{q_{L,n\lambda,w}^2} \cdot \left(T_{w^{-1}e^{-n\lambda}}^L \cdot (T_{-n\lambda}^L)^{-1}\right)^{-1} \\ &= \frac{q_{L,w}}{\mu_{U_P\cap L}(w^{-1})^2} \cdot \left(T_{ma^{-n}}^L \cdot (T_{a^{-n}}^L)^{-1}\right)^{-1} \\ &= \frac{q_{L,w}}{\mu_{U_P\cap L}(w^{-1})^2} \cdot \left((T_{a^{-n}}^L)^{-1} \cdot T_{a^{-n}m}^L\right)^{-1}. \end{aligned}$$

This expression does not depend on n by the proof of Proposition 2.29. By extension of scalars we obtain an R-linear map $\overline{\xi}_{L,M}^{G,*} : \overline{\mathcal{H}}_R(M,G) \to \overline{\mathcal{H}}_R(L,G)$. In order to show that $\overline{\xi}_{L,M}^{G,*}$ is an R-algebra homomorphism we will give a different presentation.

Lemma 4.34. *Let* **M** *be a Levi subgroup in* **G**.

(i) The R-linear map $\delta^{M,G}\colon \mathcal{H}_R(M,G) \to \overline{\mathcal{H}}_R(M,G)$ given by $\tau_w^{M,G} \mapsto \overline{\tau}_w^{M,G}$ is an isomorphism of R-algebras. We have $\delta^{M,G}(\tau_w^{M,G,*}) = \overline{\tau}_w^{M,G,*}$ for all $w \in W_M(1)$.

(ii) The R-linear map $\zeta^{M,G}\colon \mathcal{H}_R(M,G) \to \overline{\mathcal{H}}_R(M,G)$ given by $\tau_w^{M,G} \mapsto \overline{\tau}_{w^{-1}}^{M,G}$ is an anti-isomorphism of R-algebras.

Proof. (i) We may assume $R = \mathbb{Z}$. We consider the homomorphism $\overline{\delta}\colon M \to (\mathbb{Z}[p^{-1}])^\times$ given by $m \mapsto \mu_{U_P}(m^{-1}) \cdot \mu_{U_P}(m)^{-1}$; notice that this is not just an anti-homomorphism, since $(\mathbb{Z}[p^{-1}])^\times$ is abelian. By Lemma 2.12 it defines an algebra automorphism $\overline{\delta}\mathbb{1}\colon \mathcal{H}_{\mathbb{Z}[p^{-1}]}(M) \to \mathcal{H}_{\mathbb{Z}[p^{-1}]}(M)$ given by $T_m^M \mapsto \overline{\delta}(m) \cdot T_m^M$. It satisfies $\overline{\delta}\mathbb{1}\big(\mu_{U_P}(m)T_m^M\big) = \mu_{U_P}(m^{-1})T_m^M$ and hence carries $\operatorname{Im}\Theta_{M,\mathbb{Z}}^P$ onto $\operatorname{Im}\overline{\Theta}_{M,\mathbb{Z}}^P$. Clearly, we have $\delta^{M,G} = \overline{\delta}\mathbb{1}\big|_{\mathcal{H}_\mathbb{Z}(M,G)}$, whence the first assertion in (i). The last assertion follows from the fact that, for $w \in W_M(1)$, $\tau_w^{M,G,*}$ is the unique element in $\mathcal{H}_\mathbb{Z}(M,G)$ with

$$\tau_{w^{-1}}^{M,G} \cdot \tau_w^{M,G,*} = \tau_w^{M,G,*} \cdot \tau_{w^{-1}}^{M,G} = q_w,$$

and $\overline{\tau}_w^{M,G,*}$ is the unique element in $\overline{\mathcal{H}}_\mathbb{Z}(M,G)$ satisfying

$$\overline{\tau}_{w^{-1}}^{M,G} \cdot \overline{\tau}_w^{M,G,*} = \overline{\tau}_w^{M,G,*} \cdot \overline{\tau}_{w^{-1}}^{M,G} = q_w.$$

(ii) We may assume $R = \mathbb{Z}$. The anti-automorphism ζ_M on $\mathcal{H}_\mathbb{Z}(M)$ given by $T_m^M \mapsto T_{m^{-1}}^M$ satisfies $\zeta_M\big(\mu_{U_P}(m)T_m^M\big) = \mu_{U_P}(m)T_{m^{-1}}^M$, and hence carries $\operatorname{Im}\Theta_{M,\mathbb{Z}}^P$ onto $\operatorname{Im}\overline{\Theta}_{M,\mathbb{Z}}^P$. Since $\zeta^{M,G} = \zeta_M\big|_{\mathcal{H}_\mathbb{Z}(M,G)}$, this gives (ii).

□

Lemma 4.35. *Let* **M**, **L**, *and* **L**$'$ *be Levi subgroups in* **G** *with* **M** $\subseteq$ **L** $\subseteq$ **L**$'$.

(i) We have $\overline{\xi}_{L,M}^{G,*} = \delta^{L,G} \circ \xi_{L,M}^G \circ (\delta^{M,G})^{-1}$. In particular, $\overline{\xi}_{L,M}^{G,*}$ is an R-algebra homomorphism.

(ii) We have $\overline{\xi}_{L,M}^{L',*} \circ \overline{\theta}_M^{L',G} = \overline{\theta}_L^{L',G} \circ \overline{\xi}_{L,M}^{G,*}$, i. e. we have a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{H}}_R(M,G) \arrow[r, "\overline{\theta}_M^{L',G}"] \arrow[d, "\overline{\xi}_{L,M}^{G,*}"'] & \overline{\mathcal{H}}_R(M,L') \arrow[d, "\overline{\xi}_{L,M}^{L',*}"] \\
\overline{\mathcal{H}}_R(L,G) \arrow[r, "\overline{\theta}_L^{L',G}"'] & \overline{\mathcal{H}}_R(L,L').
\end{tikzcd}$$

(iii) We have $\overline{\xi}_{L',L}^{G,*} \circ \overline{\xi}_{L,M}^{G,*} = \overline{\xi}_{L',M}^{G,*}$, i. e. we have a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{H}}_R(M,G) \arrow[r, "\overline{\xi}_{L,M}^{G,*}"] \arrow[dr, "\overline{\xi}_{L',M}^{G,*}"'] & \overline{\mathcal{H}}_R(L,G) \arrow[d, "\overline{\xi}_{L',L}^{G,*}"] \\
& \overline{\mathcal{H}}_R(L',G).
\end{tikzcd}$$

Proof. We fix a strictly negative element $\lambda \in \Lambda(1)$.

First, we show $\overline{\xi}^{G,*}_{L,M} \circ \delta^{M,G} = \delta^{L,G} \circ \xi^{G}_{L,M}$. We may assume $R = \mathbb{Z}$. Let $w \in W_M(1)$ be arbitrary. Take $n \in \mathbb{N}$ such that $e^{n\lambda}w \in W_{M^-}(1)$. Recall that we have $T^L_v \cdot T^{L,*}_{v^{-1}} = T^{L,*}_{v^{-1}} \cdot T^L_v = q_{L,v}$ by Proposition 2.17 (i). Inside $\overline{\mathcal{H}}_{\mathbb{Z}[p^{-1}]}(L)$ we compute

$$
\begin{aligned}
(\overline{\xi}^{G,*}_{L,M} \circ \delta^{M,G})(\tau^{M,G,*}_w) &= \overline{\xi}^{G,*}_{L,M}(\overline{\tau}^{M,G,*}_w) = \mu_{U_P}(w^{-1}) \cdot (T^{L,*}_{n\lambda})^{-1} \cdot T^{L,*}_{e^{n\lambda}w} \\
&= \mu_{U_P}(w^{-1}) \cdot \frac{\mu_{U_{P_L}}(-n\lambda)}{\mu_{U_{P_L}}(n\lambda)} \cdot \frac{\mu_{U_{P_L}}(e^{n\lambda}w)}{\mu_{U_{P_L}}(w^{-1}e^{-n\lambda})} \cdot \delta^{L,G}\left((T^{L,*}_{n\lambda})^{-1} \cdot T^{L,*}_{e^{n\lambda}w}\right) \\
&= \mu_{U_P}(w^{-1}) \cdot \frac{\mu_{U_{P_L}}(w)}{\mu_{U_{P_L}}(w^{-1})} \cdot \frac{q_{L,e^{n\lambda}w}}{q_{L,n\lambda}} \cdot \delta^{L,G}\left(\left(T^L_{w^{-1}e^{-n\lambda}} \cdot (T^L_{-n\lambda})^{-1}\right)^{-1}\right) \\
&= \mu_{U_P}(w^{-1})^2 \cdot \frac{\mu_{U_{P_L}}(w)}{\mu_{U_{P_L}}(w^{-1})} \cdot \frac{q_{L,w}}{q^2_{L,n\lambda,w}} \cdot \delta^{L,G}\left(\left(\xi^G_{L,M}(\tau^{M,G}_{w^{-1}})\right)^{-1}\right) \\
&= \frac{\mu_{U_P}(w^{-1})^2 \cdot \mu_{U_{P_L}}(w) \cdot q_{L,w}}{\mu_{U_{P_L}}(w^{-1}) \cdot \mu_{U_P \cap L}(w^{-1})^2} \cdot (\delta^{L,G} \circ \xi^G_{L,M})\left((\tau^{M,G}_{w^{-1}})^{-1}\right) \\
&= (\delta^{L,G} \circ \xi^G_{L,M})\left(q_w \cdot (\tau^{M,G}_{w^{-1}})^{-1}\right) = (\delta^{L,G} \circ \xi^G_{L,M})(\tau^{M,G,*}_w).
\end{aligned}
$$

The fourth equality uses that $v \mapsto \mu_{U_{P_L}}(v) \cdot \mu_{U_{P_L}}(v^{-1})^{-1}$ is a group homomorphism. The sixth equality follows from Lemma 4.13 using $q_{L,n\lambda,w} = q_{L,w,n\lambda} = q_{L,-n\lambda,w^{-1}}$ and that $-n\lambda$ is strictly positive. To see that the seventh equality holds, notice that $\mu_{U_P}(w^{-1}) = \mu_{U_P\cap L}(w^{-1})\mu_{U_{P_L}}(w^{-1})$ by Lemma 4.19 and $\mu_{U_{P_L}}(w^{-1})\mu_{U_{P_L}}(w)q_{L,w} = q_w$ by Proposition 4.16 (i). The last equality follows from (4.5.11). Together with Lemma 4.34 (i) this proves (i).

Now, (iii) follows from (i) and Lemma 4.23 (iii). It remains to show (ii). We may assume that $R = \mathbb{Z}$, so that $\overline{\theta}^{L',G}_M$ and $\overline{\theta}^{L',G}_L$ are just inclusions. Let $w \in W_M(1)$ and take $n \in \mathbb{N}$ such that $e^{n\lambda}w \in W_{M^-}(1)$. Using Lemma 4.19 we compute

$$
\begin{aligned}
(\overline{\xi}^{L',*}_{L,M} \circ \overline{\theta}^{L',G}_M)(\overline{\tau}^{M,G,*}_w) &= \mu_{U_{P_{L'}}}(w^{-1}) \cdot \overline{\xi}^{L',*}_{L,M}(\overline{\tau}^{M,L',*}_w) \\
&= \mu_{U_{P_{L'}}}(w^{-1})\mu_{U_P\cap L'}(w^{-1}) \cdot (T^{L,*}_{n\lambda})^{-1} \cdot T^{L,*}_{e^{n\lambda}w} \\
&= \mu_{U_P}(w^{-1}) \cdot (T^{L,*}_{n\lambda})^{-1} \cdot T^{L,*}_{e^{n\lambda}w} \\
&= (\overline{\theta}^{L',G}_L \circ \overline{\xi}^G_{L,M})(\overline{\tau}^{M,G,*}_w).
\end{aligned}
$$

This proves (ii). □

Remark 4.36. (a) We have $\zeta^{M,L} \circ \theta^{L,G}_M = \overline{\theta}^{L,G}_M \circ \zeta^{M,G}$. In general, the diagrams

$$
\begin{tikzcd}
\mathcal{H}_R(M,G) \arrow[r, "\theta^{L,G}_M"] \arrow[d, "\delta^{M,G}"'] & \mathcal{H}_R(M,L) \arrow[d, "\delta^{M,L}"] \\
\overline{\mathcal{H}}_R(M,G) \arrow[r, "\overline{\theta}^{L,G}_M"'] & \overline{\mathcal{H}}_R(M,L)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{H}_R(M,G) \arrow[r, "\xi^G_{L,M}"] \arrow[d, "\zeta^{M,G}"'] & \mathcal{H}_R(L,G) \arrow[d, "\zeta^{L,G}"] \\
\overline{\mathcal{H}}_R(M,G) \arrow[r, "\overline{\xi}^{G,*}_M"'] & \overline{\mathcal{H}}_R(L,G)
\end{tikzcd}
$$

do *not* commute.

(b) If p is invertible in R, the equality $\overline{\xi}^{G,*}_{L,M} = \delta^{L,G} \circ \xi^G_{L,M} \circ (\delta^{M,G})^{-1}$ was essentially proved in [OV18, Lem. 2.21].[7]

4.7. Remarks on $H_R(I_P(1), P)$-modules

In this section only we consider left modules instead of right modules over the parabolic Hecke algebra $H_R\big(I_P(1), P\big)$. This does not make a difference since the anti-automorphism ζ_P on $H_R\big(I_P(1), P\big)$ (see Proposition 2.10) identifies right modules with left modules.

The following result is an observation of Schneider on [HT01, Lem. I.2.1].

Lemma 4.37. *Let $\mathcal{M}$ be a smooth P-module such that $Z(M)$ acts locally finitely on $\mathcal{M}$. Then the restricted action of U_P on $\mathcal{M}$ is trivial.*

Proof. Let $x \in \mathcal{M}$ and $u \in U_P$ be arbitrary. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a finitely generated $Z(M)$-stable R-submodule containing x. Since $\mathcal{M}$ is smooth, there exists a compact open subgroup H of P such that $hy = y$ for all $h \in H$ and $y \in \mathcal{M}_0$. Notice that $H \cap U_P$ is a compact open subgroup of U_P. Let $a \in Z(M)$ be a strictly positive element. Then there exists $n \in \mathbb{N}$ with $a^n u a^{-n} \in H \cap U_P$. Now, we have $a^n x \in \mathcal{M}_0$ and hence $ux = a^{-n} \cdot (a^n u a^{-n}) \cdot a^n x = a^{-n} \cdot a^n x = x$. This proves the lemma. ◻

It follows from Lemma 4.37 that any smooth P-module, for which the restricted action of M is admissible, actually factors through the projection $P \twoheadrightarrow M$. This leads to the question whether there is an analogous result for $H_R\big(I_P(1), P\big)$-modules.

More concretely, we may ask: let $\mathcal{M}$ be an R-module and $\rho\colon H_R\big(I_P(1), P\big) \to \mathrm{End}_R(\mathcal{M})$ an R-algebra homomorphism. Does this map factor through the homomorphism $\Theta^P_{M,R}\colon H_R\big(I_P(1), P\big) \to \mathcal{H}_R(M)$?

A necessary condition is the following: given a strictly positive element $a \in Z(M)$, we have $\Theta^P_{M,R}\big((a)_{I_P(1)}\big) = T^M_a \in \mathcal{H}_R(M)^\times$ and hence the map $\rho\big((a)_{I_P(1)}\big)$ has to be invertible. We will show that this condition is also sufficient.

We fix a strictly positive element $a \in Z(M)$.

Definition 4.38. Let $\mathcal{M}$ be an $H_R\big(I_P(1), P\big)$-module. We define the *radical* of $\mathcal{M}$ to be the R-submodule $\mathrm{Rad}(\mathcal{M}) \coloneqq \mathrm{Rad}_a(\mathcal{M}) \coloneqq \Big\{x \in \mathcal{M} \,\Big|\, \exists n \in \mathbb{N}\colon (a)^n_{I_P(1)} x = 0\Big\}$.

Lemma 4.39. *Let $\mathcal{M}$ be an $H_R\big(I_P(1), P\big)$-module.*

(i) Let $g \in P$. There exists $n \in \mathbb{N}$ such that

$$(a)^n_{I_P(1)} \cdot (g)_{I_P(1)} = \nu_M(g)\mu_{U_P}(g) \cdot (a^n g_M)_{I_P(1)} \in C(a), \tag{4.7.1}$$

where $C(a)$ is the centralizer of $(a)_{I_P(1)}$ in $H_R\big(I_P(1), P\big)$.

[7] It should be noted that there is a mistake in the formulation and proof of [OV18, Lem. 2.21]. In the claimed formula "$\delta_P(m)$" should be replaced by "$\delta_P(m^{-1})$". The mistake in the proof results from the false claim $(\tau^*_w)^{-1} = \tau_{w^{-1}} q_w$; it should be $(\tau^*_w)^{-1} = \tau_{w^{-1}} q_w^{-1}$.

(ii) $\mathrm{Rad}(\mathcal{M})$ *is an* $H_R\big(I_P(1), P\big)$*-submodule of* $\mathcal{M}$.

(iii) The definition of $\mathrm{Rad}_a(\mathcal{M})$ *does not depend on the choice of* a.

Proof. (i) Notice that $(a)^n_{I_P(1)} = \big(I_P(1)a^n\big)$ for all $n \in \mathbb{N}$, as a is central and positive. Let $g \in P$. We can write

$$(g)_{I_P(1)} = \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} \big(I_P(1) g u_s h_j m_i\big)$$

inside $X_R\big(I_P(1), P\big)$ for certain $u_s \in I_{U_P}$, $h_j \in \big(I_M(1)\big)_{(g_M)}$, and $m_i \in I_M(1)$ by Proposition 4.2. As a is strictly positive, there exists $n \in \mathbb{N}$ such that $a^n \cdot (g_U u_s)^{g_M^{-1}} \cdot a^{-n} \in I_{U_P}$ for all $s = 1, \dots, \mu_{U_P}(g)$, and with $a^n g_M \in M^+$. As a^n is central, we have

$$\begin{aligned} a^n g u_s h_j m_i &= a^n g_M g_U u_s h_j m_i \\ &= a^n \cdot (g_U u_s)^{g_M^{-1}} \cdot a^{-n} \cdot h_j^{g_M^{-1}} a^n g_M m_i \in I_P(1) a^n g_M m_i. \end{aligned}$$

Applying Proposition 4.2 to $a^n g_M$ then gives

$$\begin{aligned} (a)^n_{I_P(1)} \cdot (g)_{I_P(1)} &= \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} \big(I_P(1) a^n g u_s h_j m_i\big) \\ &= \nu_M(g)\mu_{U_P}(g) \cdot \sum_{i=1}^{\mu_M(g_M)} \big(I_P(1) a^n g_M m_i\big) \\ &= \nu_M(g)\mu_{U_P}(g) \cdot (a^n g_M)_{I_P(1)}, \end{aligned}$$

because we have $\nu_M(a^n g_M) = \mu_{U_P}(a^n g_M) = 1$ and $\mu_M(a^n g_M) = \mu_M(g_M)$ (again using that a is central). By Proposition 4.9 the element $(a^n g_M)_{I_P(1)}$ centralizes $(a)_{I_P(1)}$.

(ii) This is an immediate consequence of (i).

(iii) Let $a_0 \in Z(M)$ be another strictly positive element. By symmetry it suffices to show $\mathrm{Rad}_a(\mathcal{M}) \subseteq \mathrm{Rad}_{a_0}(\mathcal{M})$. Let $x \in \mathrm{Rad}_a(\mathcal{M})$ and let $n \in \mathbb{N}$ with $(a)^n_{I_P(1)} x = 0$. As a_0 is strictly positive, there exists $n_0 \in \mathbb{N}$ such that $m := a_0^{n_0} a^{-n} \in M^+$. As m is central and positive, we have $(m)_{I_P(1)} = \big(I_P(1)m\big)$. We compute

$$(a_0)^{n_0}_{I_P(1)} x = (m a^n)_{I_P(1)} x = (m)_{I_P(1)} \cdot (a)^n_{I_P(1)} x = 0.$$

Hence, $x \in \mathrm{Rad}_{a_0}(\mathcal{M})$. The claim follows.

□

Proposition 4.40. *Let* $\mathcal{M}$ *be an* $H_R\big(I_P(1), P\big)$*-module. The* $H_R\big(I_P(1), P\big)$*-module structure of* $\mathcal{M}/\mathrm{Rad}(\mathcal{M})$ *factors through* $H_R\big(I_P(1), P\big) \twoheadrightarrow \mathcal{H}_R(M, G)$.

In particular, if the action of $(a)_{I_P(1)}$ *on* $\mathcal{M}$ *is invertible, then* $\mathcal{M}$ *is naturally an* $\mathcal{H}_R(M)$*-module.*

Proof. It is clear that $\mathrm{Rad}(\mathcal{M}/\mathrm{Rad}(\mathcal{M})) = \{0\}$ so that the action of $(a)_{I_P(1)}$ on $\mathcal{M}/\mathrm{Rad}(\mathcal{M})$ is injective. The kernel of the map

$$t\colon H_R\big(I_P(1), P\big) \longrightarrow \mathcal{H}_R(M, G), \quad (g)_{I_P(1)} \longmapsto \frac{\nu_M(g)\mu_{U_P}(g)}{\mu_{U_P}(g_M)} \cdot \tau^{M,G}_{g_M}$$

is generated by $\left\{(g)_{I_P(1)} - \frac{\nu_M(g)\mu_{U_P}(g)}{\mu_{U_P}(g_M)} \cdot (g_M)_{I_P(1)} \,\middle|\, g \in P\right\}$. Let $g \in P$. Then we have

$$(a)^n_{I_P(1)} \cdot \left((g)_{I_P(1)} - \frac{\nu_M(g)\mu_{U_P}(g)}{\mu_{U_P}(g_M)} \cdot (g_M)_{I_P(1)}\right) = 0$$

for some $n \in \mathbb{N}$ by Lemma 4.39 (i). This shows $\mathrm{Ker}(t)\mathcal{M} \subseteq \mathrm{Rad}(\mathcal{M})$. Thus, $\mathcal{M}/\mathrm{Rad}(\mathcal{M})$ is naturally an $\mathcal{H}_R(M, G)$-module.

Now, assume that $(a)_{I_P(1)}$ acts invertibly on $\mathcal{M}$. Then $\mathrm{Rad}(\mathcal{M}) = \{0\}$. By what we have just shown $\mathcal{M}$ is naturally an $\mathcal{H}_R(M, G)$-module. Since $t\big((a)_{I_P(1)}\big) = \tau^{M,G}_a$ and $\mathcal{H}_R(M)$ is the localization of $\mathcal{H}_R(M, G)$ at $\tau^{M,G}_a$, the last statement follows. □

Lemma 4.41. *Consider the multiplicatively closed set* $\mathcal{S}_a := \left\{(a)^n_{I_P(1)} \,\middle|\, n \in \mathbb{Z}_{\geq 0}\right\}$. *It has the following properties:*

(i) For each $x \in H_R\big(I_P(1), P\big)$ *and* $n \in \mathbb{N}$ *we have* $\mathcal{S}_a x \cap H_R\big(I_P(1), P\big)(a)^n_{I_P(1)} \neq \emptyset$. *This means that* $\mathcal{S}_a$ *is* left permutable.

(ii) Let $x \in H_R\big(I_P(1), P\big)$. *If* $x(a)^n_{I_P(1)} = 0$ *for some* $n \in \mathbb{N}$, *then also* $(a)^m_{I_P(1)}x = 0$ *for some* $m \in \mathbb{N}$. *This means that* $\mathcal{S}_a$ *is* left reversible.

Hence, $\mathcal{S}_a$ *is a* left denominator set *and we may form the left ring of fractions (cf. [Lam99, §10A])*

$$\mathcal{S}_a^{-1} H_R\big(I_P(1), P\big).$$

The kernel of the natural map $H_R\big(I_P(1), P\big) \to \mathcal{S}_a^{-1} H_R\big(I_P(1), P\big)$ *coincides with the radical of* $H_R\big(I_P(1), P\big)$.

Proof. As the elements in $\mathcal{S}_a$ commute pairwise, it is easy to see that it suffices to check (i) and (ii) for elements of the form $(g)_{I_P(1)}$, $g \in P$.

Given $g \in P$ and $n \in \mathbb{N}$, Lemma 4.39 (i) guarantees the existence of some $m \in \mathbb{N}$ with

$$(a)^{n+m}_{I_P(1)} \cdot (g)_{I_P(1)} = \nu_M(g)\mu_{U_P}(g)(a^m g_M)_{I_P(1)} \cdot (a)^n_{I_P(1)}.$$

Therefore, (i) is satisfied. Similarly, if $(g)_{I_P(1)}(a)^n_{I_P(1)} = 0$, we find, by again invoking Lemma 4.39 (i), some $m \in \mathbb{N}$ with $(a)^m_{I_P(1)}(g)_{I_P(1)} \in C(a)$, and hence

$$(a)^{n+m}_{I_P(1)}(g)_{I_P(1)} = (a)^n_{I_P(1)} \cdot (a)^m_{I_P(1)}(g)_{I_P(1)} = (a)^m_{I_P(1)}(g)_{I_P(1)}(a)^n_{I_P(1)} = 0.$$

Thus, $\mathcal{S}_a$ is left reversible. According to [Lam99, §10A] the left ring of fractions $\mathcal{S}_a^{-1} H_R\big(I_P(1), P\big)$ exists and the kernel of the corresponding localization map is

$$\left\{x \in H_R\big(I_P(1), P\big) \,\middle|\, (a)^n_{I_P(1)}x = 0 \text{ for some } n \in \mathbb{N}\right\} = \mathrm{Rad}\big(H_R\big(I_P(1), P\big)\big). \qquad \square$$

Proposition 4.42. *The map $\Theta^P_{M,R} \colon H_R\big(I_P(1), P\big) \to \mathcal{H}_R(M)$ induces an R-algebra isomorphism $\mathcal{S}_a^{-1} H_R\big(I_P(1), P\big) \cong \mathcal{H}_R(M)$.*

Proof. As $\operatorname{Im}\Theta^P_{M,R}$ contains $\mathcal{H}_R(M^+)$, it follows that $\mathcal{H}_R(M)$ is the localization of $\operatorname{Im}\Theta^P_{M,R}$ at $\Theta^P_{M,R}\big((a)_{I_P(1)}\big) = T^M_a$. Hence, the induced map $\mathcal{S}_a^{-1} H_R\big(I_P(1), P\big) \to \mathcal{H}_R(M)$ is surjective. Its kernel is just $\mathcal{S}_a^{-1}\operatorname{Ker}\Theta^P_{M,R}$, which is zero, since by Proposition 4.40 we have $\operatorname{Ker}\Theta^P_{M,R} \cdot \mathcal{S}_a^{-1} H_R\big(I_P(1), P\big) \subseteq \operatorname{Rad}\big(\mathcal{S}_a^{-1} H_R\big(I_P(1), P\big)\big) = \{0\}$. Hence, the above map is an isomorphism. □

Corollary 4.43. *We have* $\operatorname{Ker}\Theta^P_{M,R} = \operatorname{Rad}\big(H_R\big(I_P(1), P\big)\big)$.

Proof. This follows immediately from Lemma 4.41 and Proposition 4.42. □

5. Hecke polynomials

We keep the notation of section 4. Recall that $\mathbf{G}$ is a connected reductive group over a local field F, and K is a special maximal parahoric subgroup of $G = \mathbf{G}(F)$, i. e. a subgroup of the form $K_{\{\varphi_0\}}$ (cf. Definition 1.42) for some special point φ_0 (cf. Theorem 1.22) in the apartment $\mathscr{A}$ (1.4.1) of G attached to the maximal F-split torus $\mathbf{T}$. Given a subset $X \subseteq G$, we will write $K_X := K \cap X$. Let Σ be the reduced root system associated with $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ (see Proposition 1.32). We fix a minimal parabolic subgroup $\mathbf{B}$ with Levi decomposition $\mathbf{B} = \mathbf{Z}\mathbf{U}$, where $\mathbf{Z}$ is the centralizer of $\mathbf{T}$ in $\mathbf{G}$. Then $\mathbf{B}$ determines a choice of positive roots Φ^+ and Σ^+, and hence also a basis Δ of Σ. Parabolic subgroups are always assumed to contain $\mathbf{B}$ if not said otherwise.

Recall also that $W_0 = N/Z$ (resp. $W = N/K_Z$) is the finite Weyl group (resp. Iwahori-Weyl group). We have a decomposition $W = \Lambda \rtimes W_0$, where $\Lambda = Z/K_Z$ is a finitely generated abelian group (1.6.4). In particular, W_0 acts on Λ via $w(\lambda) := we^{\lambda}w^{-1}$ for $w \in W_0$, $\lambda \in \Lambda$. The map $\nu \colon Z \to V = \big(X_*(T)/X_*(C)\big) \otimes_{\mathbb{Z}} \mathbb{R}$ (1.3.6) is W_0-equivariant and factors through Λ. Let Z^+ be the monoid of positive elements in Z (Definition 1.60). Its image in Λ coincides with Λ_{Z^+}, the monoid of elements $\lambda \in \Lambda$ satisfying $\langle \alpha, \nu(\lambda) \rangle \leq 0$ for all $\alpha \in \Sigma^+$ (1.8.3). In fact, Z^+ is the preimage of Λ_{Z^+} under the projection map $Z \twoheadrightarrow \Lambda$. Further, the monoid of negative elements $Z^- = (Z^+)^{-1}$ is the preimage of the monoid Λ_{Z^-} of Λ consisting of those elements $\lambda \in \Lambda$ with $\langle \alpha, \nu(\lambda) \rangle \geq 0$ for all $\alpha \in \Sigma^+$. For $x, y \in V$ we write $x \leq y$ if $y - x$ is a linear combination of simple coroots with non-negative coefficients, or, equivalently, if $\langle \varpi_\alpha, y - x \rangle \geq 0$ for all $\alpha \in \Delta$, where $\{\varpi_\alpha \mid \alpha \in \Delta\}$ is the dual basis of Δ in V^* with respect to the W_0-invariant inner product $(\,\cdot\,,\cdot\,)$.

Let R be a commutative ring with 1. In sections 5.4 and 5.5 we will restrict to the case where p is invertible in R.

5.1. Generators and relations for the parabolic Hecke algebra of $\mathrm{GL}_2(F)$

Before we develop the general theory let us investigate the parabolic Hecke algebra $H_R(K_B, B)$, where B is the subgroup of upper triangular matrices in $\mathrm{GL}_2(F)$ and $K_B = K \cap B$ is the subgroup of matrices in B with entries in O_F. We fix a uniformizer $\pi \in O_F$ and let q be the cardinality of the residue field κ_F of F. Let $\omega \colon F \to \mathbb{Z} \cup \{\infty\}$ be the valuation normalized by $\omega(\pi) = 1$. Let $\mathcal{R}$ be a system of representatives for κ_F in O_F with $0 \in \mathcal{R}$. We denote by $\mathcal{R}_B := \left\{\sum_{i=1}^n a_i \pi^{-i} \,\middle|\, n \in \mathbb{N}, a_i \in \mathcal{R}\right\}$ a system of representatives for F/O_F.

The results in this section are essentially contained in Vienney's thesis [Vie12, pp. 102 f.], yet Theorem 5.2 was only alluded to without proof.

Lemma 5.1. *The set B decomposes as*

$$B = \bigsqcup_{\substack{a,b,c \in \mathbb{Z} \\ b \leq \min\{a,c\}}} K_B \begin{pmatrix} \pi^a & \pi^b \\ 0 & \pi^c \end{pmatrix} K_B,$$

and we have for all $a, b, c \in \mathbb{Z}$ with $b \leq \min\{a, c\}$ the decomposition

$$K_B\begin{pmatrix} \pi^a & \pi^b \\ 0 & \pi^c \end{pmatrix}K_B = \begin{cases} \bigsqcup_{\substack{\beta\in\mathcal{R}_B\pi^c \\ \omega(\beta)=b}} K_B\begin{pmatrix} \pi^a & \beta \\ 0 & \pi^c \end{pmatrix}, & \text{if } b < \min\{a,c\}; \\ K_B\begin{pmatrix} \pi^a & 0 \\ 0 & \pi^c \end{pmatrix}K_B = \bigsqcup_{\substack{\beta\in\mathcal{R}_B\pi^c \\ \omega(\beta)\geq a}} K_B\begin{pmatrix} \pi^a & \beta \\ 0 & \pi^c \end{pmatrix}, & \text{if } b = \min\{a,c\}. \end{cases}$$

Proof. Let $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in B$. Write $\alpha = \alpha_0\pi^a$, $\gamma = \gamma_0\pi^c$ with $\alpha_0, \gamma_0 \in O_F^\times$, $a, c \in \mathbb{Z}$, and $\frac{\beta}{\alpha_0\pi^c} = \beta' + x$ with $\beta' \in \mathcal{R}_B$ and $x \in O_F$. Then $\beta' = 0$ or $\omega(\beta) = \omega(\beta'\pi^c)$, and

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha_0 & \alpha_0 x \\ 0 & \gamma_0 \end{pmatrix} \cdot \begin{pmatrix} \pi^a & \beta'\pi^c \\ 0 & \pi^c \end{pmatrix} \in K_B\begin{pmatrix} \pi^a & \beta'\pi^c \\ 0 & \pi^c \end{pmatrix}.$$

Given $a, a', c, c' \in \mathbb{Z}$ and $\beta, \beta' \in \mathcal{R}_B$ with

$$\begin{pmatrix} \pi^a & \beta\pi^c \\ 0 & \pi^c \end{pmatrix} \cdot \begin{pmatrix} \pi^{a'} & \beta'\pi^{c'} \\ 0 & \pi^{c'} \end{pmatrix}^{-1} = \begin{pmatrix} \pi^{a-a'} & \beta\pi^{c-c'} - \beta'\pi^{a-a'} \\ 0 & \pi^{c-c'} \end{pmatrix} \in K_B,$$

we deduce $a = a'$, $c = c'$, and then $\beta - \beta' \in O_F$, i. e. $\beta = \beta'$. Therefore, B is the disjoint union of the right cosets $K_B\begin{pmatrix} \pi^a & \beta\pi^c \\ 0 & \pi^c \end{pmatrix}$, where $a, c \in \mathbb{Z}$ and $\beta \in \mathcal{R}_B$.

Let $a, c \in \mathbb{Z}$. Take any $\beta = \beta_0 \cdot \pi^{\omega(\beta)} \in \mathcal{R}_B\pi^c$ with $\beta \neq 0$ and $\beta_0 \in O_F^\times$. If $\omega(\beta) < \min\{a, c\}$, then

$$\begin{pmatrix} \pi^a & \beta \\ 0 & \pi^c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta_0^{-1} \end{pmatrix} \cdot \begin{pmatrix} \pi^a & \pi^{\omega(\beta)} \\ 0 & \pi^c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \beta_0 \end{pmatrix} \in K_B\begin{pmatrix} \pi^a & \pi^{\omega(\beta)} \\ 0 & \pi^c \end{pmatrix}K_B.$$

If $\omega(\beta) \geq \min\{a, c\}$ then $\omega(\beta) \geq a$, because $\omega(\beta) < c$ always holds. Hence,

$$\begin{pmatrix} \pi^a & \beta \\ 0 & \pi^c \end{pmatrix} = \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^c \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta\pi^{-a} \\ 0 & 1 \end{pmatrix} \in K_B\begin{pmatrix} \pi^a & 0 \\ 0 & \pi^c \end{pmatrix}K_B.$$

The lemma follows. □

Lemma 5.1 shows that $\left\{ (\begin{pmatrix} \pi^a & \pi^b \\ 0 & \pi^c \end{pmatrix})_{K_B} \,\middle|\, a, b, c \in \mathbb{Z}, b \leq \min\{a, c\} \right\}$ is an R-basis of $H_R(K_B, B)$.

Theorem 5.2. *Let A be the R-algebra generated by elements X_+, X_-, Z, Z^{-1} subject to the following relations:*

$$\begin{aligned} ZZ^{-1} &= Z^{-1}Z = 1, \\ ZX_+ &= X_+Z, \\ ZX_- &= X_-Z, \\ X_+X_- &= q \cdot 1. \end{aligned}$$

Then we have an isomorphism of R-algebras $\rho\colon A \to H_R(K_B, B)$ given by

$$\rho(Z) := (\pi E_2)_{K_B}, \qquad \rho(X_+) := (\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_{K_B}, \qquad \rho(X_-) := (\begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix})_{K_B},$$

where $E_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

Proof. Clearly, $(E_2)_{K_B} = (K_B E_2)$ is the unit in $H_R(K_B, B)$, and $(\pi E_2)_{K_B}$ is a central and invertible element with inverse $(\pi^{-1}E_2)_{K_B}$. Moreover, we have

$$\begin{aligned}(\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix})_{K_B} \cdot (\begin{pmatrix}\pi^{-1} & 0\\ 0 & 1\end{pmatrix})_{K_B} &= (K_B\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}) \cdot \sum_{\beta\in\mathcal{R}} (K_B\begin{pmatrix}\pi^{-1} & \beta\pi^{-1}\\ 0 & 1\end{pmatrix})\\ &= \sum_{\beta\in\mathcal{R}}(K_B\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}) = q\cdot (E_2)_{K_B}.\end{aligned}$$

This shows that ρ is a well-defined R-algebra homomorphism. From the definition of A it is clear that $\left\{X_-^n X_+^m Z^k \,\middle|\, n,m\in\mathbb{Z}_{\geq 0}, k\in\mathbb{Z}\right\}$ constitutes a basis of A as an R-module.

It is easy to see that $\rho(X_+^m) = ((\begin{smallmatrix}\pi^m & 0\\ 0 & 1\end{smallmatrix}))_{K_B}$ and $\rho(Z^k) = (\pi^k E_2)_{K_B}$ for $m\in\mathbb{Z}_{\geq 0}$ and $k\in\mathbb{Z}$. For $n\in\mathbb{Z}_{\geq 0}$ we compute

$$\begin{aligned}\rho(X_-^n) = \rho(X_-)^n &= \left(\sum_{\beta\in\mathcal{R}}(K_B\begin{pmatrix}\pi^{-1} & \beta\pi^{-1}\\ 0 & 1\end{pmatrix})\right)^n\\ &= \sum_{\beta_1,\dots,\beta_n\in\mathcal{R}}(K_B\begin{pmatrix}\pi^{-n} & \sum_{i=1}^n \beta_i\pi^{-i}\\ 0 & 1\end{pmatrix}) = (\begin{pmatrix}\pi^{-n} & 0\\ 0 & 1\end{pmatrix})_{K_B}.\end{aligned}$$

For $n,m\in\mathbb{Z}_{\geq 0}$ and $k\in\mathbb{Z}$ this gives us

$$\begin{aligned}\rho(X_-^n X_+^m Z^k) &= (\begin{pmatrix}\pi^{-n} & 0\\ 0 & 1\end{pmatrix})_{K_B}\cdot(\begin{pmatrix}\pi^{m} & 0\\ 0 & 1\end{pmatrix})_{K_B}\cdot(\pi^k E_2)_{K_B}\\ &= \sum_{\beta_1,\dots,\beta_n\in\mathcal{R}}(K_B\begin{pmatrix}\pi^{m+k-n} & \pi^k\sum_{i=1}^n\beta_i\pi^{-i}\\ 0 & \pi^k\end{pmatrix})\\ &= \sum_{b=k-n}^{\min\{k,m+k-n\}}(\begin{pmatrix}\pi^{m+k-n} & \pi^b\\ 0 & \pi^k\end{pmatrix})_{K_B}.\end{aligned}$$

For $n,m\in\mathbb{N}$, and $k\in\mathbb{Z}$ we thus have

$$\rho(X_-^nX_+^mZ^k - X_-^{n-1}X_+^{m-1}Z^k) = (\begin{pmatrix}\pi^{m+k-n} & \pi^{k-n}\\ 0 & \pi^k\end{pmatrix})_{K_B}.$$

Hence ρ maps the basis

$$\begin{aligned}\left\{X_+^mZ^k \,\middle|\, m\in\mathbb{Z}_{\geq 0}, k\in\mathbb{Z}\right\} &\cup \left\{X_-^nZ^k \,\middle|\, n\in\mathbb{N}, k\in\mathbb{Z}\right\}\\ &\cup\left\{X_-^nX_+^mZ^k - X_-^{n-1}X_+^{m-1}Z^k \,\middle|\, n,m\in\mathbb{N}, k\in\mathbb{Z}\right\}\end{aligned}$$

of A to the canonical basis of $H_R(K_B, B)$. It follows that ρ is an isomorphism of R-algebras. □

Corollary 5.3. *Write $X_+ = ((\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}))_{K_B}$ and $X_- = ((\begin{smallmatrix}\pi^{-1} & 0\\ 0 & 1\end{smallmatrix}))_{K_B}$ in $H_R(K_B, B)$. Then X_+ (resp. X_-) is a left (resp. right) zero-divisor. If q is invertible in R, then X_+ (resp. X_-) is right (resp. left) invertible.*

Proof. It follows from Theorem 5.2 that

$$X_+ \cdot (X_- X_+ - q \cdot 1) = (X_- X_+ - q \cdot 1) \cdot X_- = 0.$$

If q is invertible in R, then $X_+ \cdot q^{-1} X_- = q^{-1} X_+ \cdot X_- = 1$. □

5.2. The Satake homomorphism

We recall here the Satake homomorphism of the spherical Hecke algebra as developed in [HV15]. Afterwards, we will give a new description involving the parabolic Hecke algebra.

Definition 5.4. We call

$$H_R(K, G) \tag{5.2.1}$$

the *spherical Hecke algebra* of G. Recall that it is the algebra of K-invariants of the free R-module $X_R(K, G)$ on generators (Kg) for $Kg \in K\backslash G$. Then

$$\left\{ (g)_K := (KgK) \,\middle|\, KgK \in K\backslash G/K \right\}$$

is a basis of $H_R(K, G)$ as an R-module, and multiplication is given by a convolution product (cf. Lemma 2.6).

In the literature (e. g. [HR09], [HV15], [Her11a]) the algebra $H_R(K, G)$ is defined as the algebra $C_c^\infty(K\backslash G/K, R)$ of compactly supported, K-biinvariant functions $G \to R$ with convolution product given by

$$(f_1 * f_2)(g) = \sum_{hK \in G/K} f_1(h) \cdot f_2(h^{-1}g), \qquad \text{for } f_1, f_2 \in C_c^\infty(K\backslash G/K, R),\ g \in G.$$

It is a straightforward computation to show that the map

$$\rho_G \colon C_c^\infty(K\backslash G/K, R) \longrightarrow H_R(K, G), \quad f \longmapsto \sum_{Kg \in K\backslash G} f(g^{-1}) \cdot (Kg)$$

is an isomorphism of R-algebras.[8] Likewise, $H_R(K_Z, Z)$ is identified with the algebra $C_c^\infty(Z/K_Z, R)$ of compactly supported, K_Z-biinvariant functions $Z \to R$ with convolution product (recall that K_Z is normal in Z).

Definition 5.5. The map

$$\mathcal{S}' = \mathcal{S}'_G \colon C_c^\infty(K\backslash G/K, R) \longrightarrow C_c^\infty(Z/K_Z, R) = R[\Lambda],$$

given by

$$\mathcal{S}'(f)(z) = \sum_{u(K_U) \in U/(K_U)} f(zu), \qquad \text{for } f \in C_c^\infty(K\backslash G/K, R) \text{ and } z \in Z$$

is called the *Satake homomorphism*.

[8] The inverse element in the formula is needed to account for the transition from left cosets to right cosets.

If $R = \mathbb{C}$, the classical Satake homomorphism is defined as

$$\delta^{1/2}\mathcal{S}' \colon f \longmapsto \Big[z \mapsto \delta^{1/2}(z) \cdot \mathcal{S}'(f)(z) \Big],$$

where $\delta^{1/2} \colon z \mapsto [z(K_U)z^{-1} : K_U]^{1/2}$ is the square root of the modulus character of B, restricted to Z. Notice that $\delta(z)$ only depends on the coset $K_Z z$. Thus, δ descends to a group homomorphism $\Lambda \to \mathbb{Q}^\times$, which is again denoted by δ. The image of $\delta^{1/2}\mathcal{S}'$ is the subalgebra $\mathbb{C}[\Lambda]^{W_0}$ of W_0-invariants, where W_0 acts by conjugation on Λ. When p is not invertible in R, we cannot make use of δ. Herzig [Her11a] came up with Definition 5.5 in order to provide a Satake homomorphism for general R. It was then studied by Henniart and Vignéras in [HV15].

In order to describe the image of $\mathcal{S}'$, Henniart and Vignéras define a twisted action of W_0 on $\mathbb{Z}[p^{-1}][\Lambda]$ via

$$w * e^\lambda := \delta^{1/2}\big(\lambda - w(\lambda)\big) \cdot e^{w(\lambda)}, \quad \text{for } w \in W_0,\ \lambda \in \Lambda. \tag{5.2.2}$$

(Here, we write e^λ for the image of λ in $\mathbb{Z}[p^{-1}][\Lambda]$.) One can show that the cocycle $(w, \lambda) \mapsto \delta^{1/2}(\lambda - w(\lambda))$ takes values in $p^{\mathbb{Z}}$ [HV15, 7.12], and thus the twisted action of W_0 on $\mathbb{Z}[p^{-1}][\Lambda]$ is well-defined.

Let $\lambda \in \Lambda_{Z^-}$ and denote by $W_{0,\lambda}$ the stabilizer of λ in W_0. Then we have [HV15, 7.13]

$$S'_\lambda := \sum_{w \in W_0/W_{0,\lambda}} w * e^\lambda \in \mathbb{Z}[\Lambda],$$

where the notation "$w \in W_0/W_{0,\lambda}$" means that w runs through a system of representatives of $W_0/W_{0,\lambda}$ inside W_0.

We can now formulate the main result of [HV15].

Theorem 5.6. *Consider the Satake homomorphism $\mathcal{S}' \colon C_c^\infty(K\backslash G/K, R) \to R[\Lambda]$. Then*

(i) $\mathcal{S}'$ is injective;

(ii) the image of $\mathcal{S}'$ is a free R-module with basis $\{1 \otimes S'_\lambda \mid \lambda \in \Lambda_{Z^-}\}$. If p is invertible in R, then it coincides with $R[\Lambda]^{W_0}$, the algebra of W_0-invariant elements under the twisted action.

(iii) both $R[\Lambda]$ and $C_c^\infty(K\backslash G/K, R)$ are commutative algebras of finite type over R.

Proof. (i) and (ii) is [HV15, 7.13 Cor., 7.15 Thm.], and (iii) is [HV15, 7.16]. □

The proof of Theorem 5.6 makes crucial use of two double coset decompositions of G which we want to recall here:

Proposition 5.7. *(i) **Cartan decomposition:** The inclusion $Z \subseteq G$ induces bijections*

$$\Lambda_{Z^+} \cong K\backslash G/K \quad \text{and} \quad \Lambda_{Z^-} \cong K\backslash G/K.$$

*(ii) **Iwasawa decomposition:** The inclusion $Z \subseteq G$ induces a bijection*

$$\Lambda \cong K\backslash G/U.$$

This decomposition is often written as $G = KB = KZU = KUZ$.

Proof. (i) is [HV15, 6.3 Prop.] and (ii) follows from [HV15, 6.5]. □

Remark 5.8. Let $z \in Z^-$ and $z' \in Z$ such that the intersection $KzK \cap Kz'U$ is not empty. It is known (e. g. [HV15, 6.10 Prop.]) that this implies $\nu(z) \geq \nu(z')$, i. e. $\nu(z) - \nu(z')$ is a sum of simple coroots in V; if moreover $\nu(z) = \nu(z')$, then $K_Z z = K_Z z'$.

It is natural to ask what can be said (e. g. in terms of the valuation φ_0) about the $u \in U$ with $z'u \in KzK$. What is known is that $z' = z$ implies $u \in K_U$ (e. g. [HV15, 6.9 Prop.]). In section 5.5 we will further investigate this question.

We will now give a different description of the Satake homomorphism using the parabolic Hecke algebra. Before doing so, we will translate Theorem 5.6 into our context.

Lemma 5.9. *Consider the map*

$$\mathcal{S} = \mathcal{S}_G \colon H_R(K,G) \longrightarrow H_R(K_Z,Z) = R[\Lambda], \quad \sum_{i=1}^r a_i \cdot (Ku_iz_i) \longmapsto \sum_{i=1}^r a_i \cdot (z_i)_{K_Z},$$

where $u_i \in U$, $z_i \in Z$.[9] *Then the diagram*

$$\begin{array}{ccc} C_c^\infty(K\backslash G/K,R) & \xrightarrow{\mathcal{S}'} & C_c^\infty(Z/K_Z,R) \\ {\scriptstyle\rho_G}\downarrow & & \downarrow{\scriptstyle\rho_Z} \\ H_R(K,G) & \xrightarrow[\mathcal{S}]{} & H_R(K_Z,Z) \end{array}$$

is commutative. Henceforth, $\mathcal{S}$ is also called the Satake homomorphism. *Moreover, if we define the twisted action of W_0 on $H_{\mathbb{Z}[p^{-1}]}(K_Z,Z)$ by*

$$w \star e^\lambda = \delta^{1/2}\big(w(\lambda) - \lambda\big) \cdot e^{w(\lambda)}, \quad \textit{for } w \in W_0,\ \lambda \in \Lambda, \tag{5.2.3}$$

then $\rho_Z \colon C_c^\infty(Z/K_Z,\mathbb{Z}[p^{-1}]) \to H_{\mathbb{Z}[p^{-1}]}(K_Z,Z)$ is W_0-equivariant.

Proof. We fix a representing system $\Gamma \subseteq Z$ of the coset space $K_Z\backslash Z$. The map

$$K_U\backslash U \times \Gamma \longrightarrow K\backslash G, \quad \big(K_U u, z\big) \longmapsto Kuz$$

is a bijection: surjectivity follows from the Iwasawa decomposition 5.7, (ii) and the fact that K_Z normalizes U. Given $u_1,u_2 \in U$, $z_1,z_2 \in \Gamma$ with $Ku_1z_1 = Ku_2z_2$, we have $u_1z_1z_2^{-1}u_2^{-1} = k \in K \cap B$. Under the projection $\mathrm{pr}_Z \colon B \twoheadrightarrow Z$ we obtain $z_1z_2^{-1} = \mathrm{pr}_Z(k) \in K_Z$, and hence $z_1 = z_2$. Therefore, $u_1u_2^{-1} = k \in K \cap U$, i. e. $K_U u_1 = K_U u_2$; this proves injectivity.

Let $f \in C_c^\infty(K\backslash G/K,R)$. We compute

$$\begin{aligned} \rho_Z\big(\mathcal{S}'(f)\big) &= \sum_{z\in\Gamma} \mathcal{S}'(f)(z^{-1}) \cdot (K_Z z) = \sum_{z\in\Gamma} \sum_{uK_U \in U/K_U} f\big(z^{-1}u\big) \cdot (K_Z z) \\ &= \sum_{z\in\Gamma} \sum_{K_U u \in K_U\backslash U} f\big((uz)^{-1}\big) \cdot (K_Z z) = \sum_{Kuz \in K\backslash G} f\big((uz)^{-1}\big) \cdot (K_Z z) \\ &= \mathcal{S}\left(\sum_{Kuz\in K\backslash G} f\big((uz)^{-1}\big) \cdot (Kuz) \right) = \mathcal{S}\big(\rho_G(f)\big). \end{aligned}$$

[9] Notice that by the Iwasawa decomposition 5.7, (ii) every right coset in $K\backslash G$ is of the form Kuz for some $u \in U$ and $z \in Z$.

Hence, $\rho_Z \circ \mathcal{S}' = \mathcal{S} \circ \rho_G$. Now let $w \in W_0$ and $\lambda \in \Lambda$. Then

$$\rho_Z\big(w * e^\lambda\big) = \rho_Z\big(\delta^{1/2}(\lambda - w(\lambda)) \cdot e^{w(\lambda)}\big) = \delta^{1/2}\big(-w(\lambda) - (-\lambda)\big) \cdot e^{-w(\lambda)} = w \star \rho_Z(e^\lambda).$$

□

Theorem 5.10. *Consider the Satake homomorphism $\mathcal{S}\colon H_R(K,G) \to R[\Lambda]$. Then*

(i) $\mathcal{S}$ is injective;

(ii) the image of $\mathcal{S}$ is a free R-module with basis $\{1 \otimes S_\lambda \mid \lambda \in \Lambda_{Z^+}\}$, where

$$S_\lambda := \sum_{w \in W_0/W_{0,\lambda}} w \star e^\lambda \in \mathbb{Z}[\Lambda], \quad \textit{for } \lambda \in \Lambda_{Z^+}.$$

If p is invertible in R, then this coincides with $R[\Lambda]^{W_0}$, the algebra of W_0-invariant elements under the twisted action.

(iii) both $R[\Lambda]$ and $H_R(K,G)$ are commutative algebras of finite type over R.

Proof. Lemma 5.9 and Theorem 5.6. □

Remark 5.11. It follows from [HV15, 7.3, Lem. 2] and a "triangular argument" that for each $\lambda \in \Lambda_{Z^+}$ the element $\mathcal{S}^{-1}(S_\lambda)$ is of the form

$$\mathcal{S}^{-1}(S_\lambda) = (z_\lambda)_K + \sum_{\substack{\mu \in \Lambda_{Z^+},\\ \nu(\lambda) < \nu(\mu)}} c_\mu \cdot (z_\mu)_K,$$

where z_μ (resp. z_λ) is a lift of $\mu \in \Lambda_{Z^+}$ (resp. λ). If $p \in R^\times$ then an analogous statement with Λ_{Z^+} replaced by Λ_{Z^-} also holds.

Example 5.12. Let p be invertible in R. Consider the case $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O}_F)$. Let $\pi \in \mathcal{O}_F$ be a prime element and denote by q the cardinality of the residue field of F. Fix a system of representatives $\mathcal{R}$ of $\mathcal{O}_F/(\pi)$ in $\mathcal{O}_F$ with $0 \in \mathcal{R}$. Let T be the subgroup of diagonal matrices in G. Then $\Lambda = K_T\backslash T \cong \mathbb{Z}^2$, and hence $H_R(K_T, T)$ is isomorphic to $R[X^{\pm 1}, Y^{\pm 1}]$ via $X \mapsto (K_T \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right))$ and $Y \mapsto (K_T \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix}\right))$. Moreover, Λ_{T^+} corresponds to the submonoid $\{(a,b) \in \mathbb{Z}^2 \,|\, a \geq b\}$ of $\mathbb{Z}^2$. The twisted action of the finite Weyl group $W_0 = \{1, s\}$ on $R[X^{\pm 1}, Y^{\pm 1}]$ is given by $sX = qY$ and $sY = q^{-1}X$. Given $a, b, c \in \mathbb{Z}$ with $a > b$, we put

$$S_{a,b} := X^a Y^b + q^{a-b} X^b Y^a \quad \text{and} \quad S_{c,c} := X^c Y^c.$$

We will now describe the Satake homomorphism and show that the $S_{a,b}$ generate its image. The Cartan decomposition reads $G = \bigsqcup_{a \geq b} K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} K$. For each $a, b \in \mathbb{Z}$, $a \geq b$ we have a decomposition

$$\begin{aligned} K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} K = & \bigsqcup_{i=0}^{\lfloor \frac{a-b}{2} \rfloor} K \begin{pmatrix} \pi^{a-i} & 0 \\ 0 & \pi^{b+i} \end{pmatrix} \\ & \sqcup \bigsqcup_{i=\lfloor \frac{a-b}{2} \rfloor + 1}^{a-b} \; \bigsqcup_{u_{a-i}, \ldots, u_{b+i-1} \in \mathcal{R}} K \begin{pmatrix} \pi^{a-i} & \sum_{j=a-i}^{b+i-1} u_j \pi^j \\ 0 & \pi^{b+i} \end{pmatrix} \end{aligned} \tag{5.2.4}$$

Let $a, b \in \mathbb{Z}$ with $a \geq b$ and write $c := \lfloor (a-b)/2 \rfloor$. Then we compute

$$\mathcal{S}\left((\begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix})_K \right) = \sum_{i=0}^{c} X^{a-i}Y^{b+i} + \sum_{i=c+1}^{a-b} q^{b-a+2i} X^{a-i}Y^{b+i} = \sum_{i=0}^{c} S_{a-i,b+i}.$$

Define a total ordering on Λ_{T^+} by letting $(a,b) > (c,d)$ if either $a-b > c-d$, or $a-b = c-d$ and $a > c$. Then it is an easy exercise to verify that $\mathcal{S}$ is injective and that the image of $\mathcal{S}$ is generated as an R-algebra by $S_{1,0}$ and $S_{1,1}^{\pm 1}$. Hence, $H_R(K,G)$ is the polynomial algebra generated by $((\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}))_K$ and $(\pi E_2)_K^{\pm 1}$.

Lemma 5.13. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be parabolic subgroups of* $\mathbf{G}$ *with* $\mathbf{P} \subseteq \mathbf{Q}$.[10] *Then we have an embedding*

$$\varepsilon_{P,Q} \colon H_R(K_Q, Q) \hookrightarrow H_R(K_P, P), \quad \sum_{i=1}^{n} a_i \cdot (K_Q g_i) \longmapsto \sum_{i=1}^{n} a_i \cdot (K_P g_i)$$

where we may choose $g_i \in P$*. Moreover, the following diagram is commutative:*

$$\begin{tikzcd} H_R(K,G) \arrow[r, hook, "\varepsilon_{Q,G}"] \arrow[dr, hook, "\varepsilon_{P,G}"'] & H_R(K_Q,Q) \arrow[d, hook, "\varepsilon_{P,Q}"] \\ & H_R(K_P,P). \end{tikzcd}$$

Proof. Compare [Gri92, pp. 2870 f.]. We need to verify the conditions (2.1.2) of Proposition 2.9. It is clear that $K_P \subseteq K_Q$ and $K_Q \cap P \subseteq K_P$. Moreover, by [HV15, 6.5] we have the Iwasawa decomposition $G = KP$, from which we deduce $Q = K_Q P$. Hence, we obtain a canonical embedding $\varepsilon_{P,Q} \colon H_R(K_Q,Q) \hookrightarrow H_R(K_P,P)$.

Given $X = \sum_{i=1}^{n} a_i \cdot (K g_i) \in H_R(K,G)$, we may choose the g_i in P, and then we have

$$\varepsilon_{P,Q}\big(\varepsilon_{Q,G}(X)\big) = \varepsilon_{P,Q}\left(\sum_{i=1}^{n} a_i \cdot (K_Q g_i) \right) = \sum_{i=1}^{n} a_i \cdot (K_P g_i) = \varepsilon_{P,G}(X).$$

This proves $\varepsilon_{P,Q} \circ \varepsilon_{Q,G} = \varepsilon_{P,G}$. □

Remark 5.14. The existence of the embedding $\varepsilon_{P,G}$ in Lemma 5.13 relies on the Iwasawa decomposition $G = KP$. Lemma 5.13 does not hold if we replace K by the Iwahori (resp. pro-p Iwahori) subgroup I (resp. $I(1)$).

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{M}\mathbf{U}_P$. Then K_M is a special maximal parahoric subgroup of M [HR09, Lem. 4.1.1], and we have $K_P = K_M K_{U_P}$ [HV15, 6.5]. Hence, for every $g \in K_P$ we have a (unique) decomposition $g = g_M g_U$, where $g_M := \mathrm{pr}_M(g) \in K_M$ and $g_U := g_M^{-1} g \in K_{U_P}$. Using this, Proposition 4.3 shows that

$$\Theta_M^P = \Theta_{M,R}^P \colon H_R(K_P,P) \longrightarrow H_R(K_M,M), \quad \sum_{i=1}^{n} a_i \cdot (K_P g_i) \longmapsto \sum_{i=1}^{n} a_i \cdot (K_M g_{i,M})$$

is a homomorphism of R-algebras. It satisfies $\Theta_M^P\big((g)_{K_P}\big) = \nu_M(g)\mu_{U_P}(g) \cdot (g_M)_{K_M}$ for all $g \in P$.

[10]We expressly permit $\mathbf{Q} = \mathbf{G}$ here.

Proposition 5.15. *(a) The Satake homomorphism factors as $\mathcal{S} = \Theta_Z^B \circ \varepsilon_{B,G}$, i. e. we have a commutative diagram*

$$\begin{tikzcd} H_R(K_B, B) \arrow[dr, "\Theta_Z^B"] & \\ H_R(K, G) \arrow[u, hook, "\varepsilon_{B,G}"] \arrow[r, "\mathcal{S}"'] & H_R(K_Z, Z). \end{tikzcd}$$

(b) Given any parabolic subgroup $\mathbf{P}$ *of* $\mathbf{G}$ *with Levi decomposition* $\mathbf{P} = \mathbf{M}\mathbf{U}_P$*, we have a commutative diagram*

$$\begin{tikzcd} H_R(K_B, B) \arrow[r, "\Theta_Z^B"] & H_R(K_Z, Z) \\ H_R(K_P, P) \arrow[u, hook, "\varepsilon_{B,P}"] \arrow[r, "\Theta_M^P"'] & H_R(K_M, M). \arrow[u, hook, "\mathcal{S}_M"'] \end{tikzcd}$$

In particular, the Satake homomorphism factors as $\mathcal{S} = \mathcal{S}_M \circ \Theta_M^P \circ \varepsilon_{P,G}$, i. e. we have a commutative diagram

$$\begin{tikzcd} H_R(K_P, P) \arrow[r, "\Theta_M^P"] & H_R(K_M, M) \arrow[d, hook, "\mathcal{S}_M"] \\ H_R(K, G) \arrow[u, hook, "\varepsilon_{P,G}"] \arrow[r, "\mathcal{S}"'] & H_R(K_Z, Z). \end{tikzcd}$$

Proof. (a) Given $X := \sum_{i=1}^r a_i \cdot (Kg_i) \in H_R(K, G)$, let $u_i \in U$, $z_i \in Z$ be such that $Kg_i = Ku_iz_i$. Using the definition of $\mathcal{S}$ in Lemma 5.9 we compute

$$\Theta_Z^B\big(\varepsilon_{B,G}(X)\big) = \Theta_Z^B\left(\sum_{i=1}^r a_i \cdot (K_B u_i z_i)\right) = \sum_{i=1}^r a_i \cdot (z_i)_{K_Z} = \mathcal{S}(X).$$

(b) Let $X = \sum_{i=1}^r a_i \cdot (K_P g_i) \in H_R(K_P, P)$. Let $u_i \in U_P$, $m_i \in M \cap U$, $z_i \in Z$ be elements such that $K_P g_i = K_P u_i m_i z_i$ for each i. Then,

$$\begin{aligned} (\Theta_Z^B \circ \varepsilon_{B,P})(X) &= \sum_{i=1}^r a_i \cdot \Theta_Z^B(K_B u_i m_i z_i) = \sum_{i=1}^r a_i \cdot (K_Z z_i) \\ &= \mathcal{S}_M\left(\sum_{i=1}^r a_i \cdot (K_M m_i z_i)\right) = (\mathcal{S}_M \circ \Theta_M^P)(X). \end{aligned}$$

Thus, $\Theta_Z^B \circ \varepsilon_{B,P} = \mathcal{S}_M \circ \Theta_M^P$. By (a) and Lemma 5.13 we conclude

$$\mathcal{S} = \Theta_Z^B \circ \varepsilon_{B,G} = \Theta_Z^B \circ \varepsilon_{B,P} \circ \varepsilon_{P,G} = \mathcal{S}_M \circ \Theta_M^P \circ \varepsilon_{P,G}. \qquad \square$$

Example 5.16. Let us return to the context of Example 5.12. By Theorem 5.2 the parabolic Hecke algebra $H_R(K_B, B)$ is generated by the elements $X_+ := ((\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}))_{K_B}$, $X_- := ((\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}))_{K_B}$, and $Z^{\pm 1}$, where $Z := (\pi E_2)_{K_B}$ is central, subject to the relation $X_+ X_- = q \cdot 1$.

The map $\Theta_T^B \colon H_R(K_B, B) \to R[X^{\pm 1}, Y^{\pm 1}]$ is given by $X_+ \mapsto X$, $X_- \mapsto qX^{-1}$, $Z \mapsto XY$. The kernel of Θ_T^B is the two-sided ideal generated by $X_- X_+ - q \cdot 1$.

By (5.2.4) we have

$$K \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} K = K \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \sqcup \bigsqcup_{u \in \mathcal{R}} K \begin{pmatrix} 1 & u \\ 0 & \pi \end{pmatrix}.$$

Hence, the embedding $\varepsilon_{B,G} \colon H_R(K, G) \to H_R(K_B, B)$ is given by $((\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}))_K \mapsto X_+ + X_- Z$ and $(\pi E_2)_K \mapsto Z$. Therefore,

$$\begin{aligned} \Theta_T^B \left(\varepsilon_{B,G} \left((\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_K \right) \right) &= \Theta_T^B (X_+ + X_- Z) = X + qX^{-1} \cdot (XY) \\ &= X + qY = \mathcal{S} \left((\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_K \right), \\ \Theta_T^B \left(\varepsilon_{B,G} \big((\pi E_2)_K \big) \right) &= \Theta_T^B (Z) = XY = \mathcal{S}\big((\pi E_2)_K \big). \end{aligned}$$

We end this section by giving another description of the twisted action (5.2.3). Recall from section 4.1 the map $\mu_U \colon B \to \mathbb{N}$, given by $\mu_U(g) = [K_U : (K_U)_{(g)}]$, where $(K_U)_{(g)} = K_U \cap g^{-1} K_U g$. By Remark 2.4 we may write the modulus character $\delta \colon B \to \mathbb{N}$ of B as

$$\delta(g) = [K_U : g^{-1} K_U g] = \mu_U(g) / \mu_U(g^{-1}).$$

In Notation 4.12 it was established that μ_U descends to a map $\Lambda \to q^{\mathbb{Z}_{\geq 0}}$, again denoted by μ_U.

Lemma 5.17. *For each $\lambda \in \Lambda$ and $w \in W_0$ we have*

$$w \star e^\lambda = \frac{\mu_U(w(\lambda))}{\mu_U(\lambda)} \cdot e^{w(\lambda)}.$$

In particular, if $\lambda \in \Lambda_{Z^+}$, the symmetrized element $S_\lambda = \sum_{w \in W_0 / W_{0,\lambda}} w \star e^\lambda$ has integral coefficients.

Proof. Let $\lambda \in \Lambda$, $w \in W_0$. It suffices to show

$$\frac{\mu_U(w(\lambda)) / \mu_U(-w(\lambda))}{\mu_U(\lambda) / \mu_U(-\lambda)} = \frac{\delta(w(\lambda))}{\delta(\lambda)} = \delta\big(w(\lambda) - \lambda\big) \overset{!}{=} \frac{\mu_U(w(\lambda))^2}{\mu_U(\lambda)^2},$$

or, equivalently,

$$\mu_U(\lambda) \mu_U(-\lambda) = \mu_U\big(w(\lambda)\big) \mu_U\big(-w(\lambda)\big). \tag{5.2.5}$$

One way to see this is to recall from Proposition 4.16, (i) that $q_\lambda = \mu_U(\lambda) \mu_U(-\lambda)$. Now, the assertion follows from $q_\lambda = q_{w(\lambda)}$ [Vig16, Prop. 5.13].

Alternatively, we can also give a direct proof of (5.2.5) (which then gives a new proof for $q_\lambda = q_{w(\lambda)}$). From Step 2 in the proof of Lemma 4.13 we know

$$\mu_U(\lambda) = \prod_{\alpha\in\Sigma^+} Q(\langle\alpha,\nu(\lambda)\rangle), \quad \text{where } Q(k) := \begin{cases} \left|U_{(\alpha,0)}/U_{(\alpha,k)}\right|, & \text{for } k \in \mathbb{Z}_{\geq 0}; \\ 1, & \text{for } k \in \mathbb{Z}_{<0}. \end{cases}$$

Using the W_0-invariance of $\langle\cdot,\cdot\rangle$ we compute

$$\begin{aligned} \mu_U\big(w(\lambda)\big)\cdot\mu_U\big(-w(\lambda)\big) &= \prod_{\alpha\in\Sigma^+} Q\big(\langle\alpha,\nu(w(\lambda))\rangle\big)\cdot\prod_{\alpha\in\Sigma^+} Q\big(\langle-\alpha,\nu(w(\lambda))\rangle\big) \\ &= \prod_{\alpha\in\Sigma} Q\big(\langle\alpha,\nu(w(\lambda))\rangle\big) = \prod_{\alpha\in\Sigma} Q\big(\langle w(\alpha),\nu(w(\lambda))\rangle\big) \\ &= \prod_{\alpha\in\Sigma} Q\big(\langle\alpha,\nu(\lambda)\rangle\big) = \mu_U(\lambda)\cdot\mu_U(-\lambda). \end{aligned}$$

The last assertion follows from $\mu_U(\lambda) = 1$ for $\lambda \in \Lambda_{Z^+}$. □

5.3. Centralizers in parabolic Hecke algebras

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{MU}_P$. We choose a strictly positive element $a_P \in M^+$ (Definition 1.60). Notice that $(a_P)_{K_P} = (K_P a_P)$ in $H_R(K_P, P)$, because a_P normalizes U_P and centralizes M. In this section we will be concerned with the centralizer algebra

$$C_P^+ := C_{H_R(K_P,P)}\big((a_P)_{K_P}\big) := \left\{X \in H_R(K_P,P) \,\middle|\, X\cdot(a_P)_{K_P} = (a_P)_{K_P}\cdot X\right\}. \quad (5.3.1)$$

Lemma 5.18. *(i) The inclusion $Z \subseteq M$ induces a bijection $\Lambda_{Z^+} \cong K_M\backslash M^+/K_M$.*

(ii) The set $\left\{(z)_{K_P} \,\middle|\, K_Z z \in \Lambda_{Z^+}\right\}$ is a basis of C_P^+ as an R-module. The restriction of Θ_M^P to C_P^+ induces an isomorphism $C_P^+ \cong H_R(K_M, M^+)$. In particular, C_P^+ is commutative.

Proof. (i) By the Cartan decomposition 5.7, (i) the inclusion $Z \subseteq M$ induces an isomorphism $\Lambda_{Z^+,M} \cong K_M\backslash M/K_M$, where

$$\Lambda_{Z^+,M} = \left\{\lambda \in \Lambda \,\middle|\, \langle\alpha,\nu(\lambda)\rangle \leq 0 \quad \text{for all } \alpha \in \Sigma_M^+\right\}.$$

As in Proposition 1.62, (ii) one shows that $\lambda \in \Lambda$ corresponds to $K_M m K_M$ for $m \in M^+$ if and only if

$$\lambda \in \Lambda_{M^+} = \left\{\lambda \in \Lambda \,\middle|\, \langle\alpha,\nu(\lambda)\rangle \leq 0 \text{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_M\right\}.$$

Now, the assertion follows from $\Lambda_{Z^+} = \Lambda_{Z^+,M} \cap \Lambda_{M^+}$.

(ii) Notice that $(z)_{K_P}$ depends only on the coset $K_Z z$ for $z \in Z$. By Proposition 4.9 we know that $\left\{(m)_{K_P} \,\middle|\, m \in M^+\right\}$ is an R-basis for C_P^+ and that the restriction of Θ_M^P to C_P^+ induces an isomorphism $C_P^+ \cong H_R(K_M, M^+)$. The rest of the assertion follows from (i). From Theorem 5.10 it follows that C_P^+ is commutative.

□

Proposition 5.19. *Let* $\mathbf{Q} = \mathbf{L}\mathbf{U}_Q$ *be another parabolic subgroup of* $\mathbf{G}$. *Let* $a_Q \in L$ *be strictly positive.*

(i) Assume $P \subseteq Q$. *Then we have*

$$\begin{aligned}\varepsilon_{P,Q}(C_Q^+) &= C_{H_R(K_P,P)}\big((a_Q)_{K_P}\big)\\ &= \big\{X \in H_R(K_P,P) \,\big|\, X \cdot (a_Q)_{K_P} = (a_Q)_{K_P} \cdot X\big\}\,.\end{aligned}$$

(ii) The element $a_{P\cap Q} \coloneqq a_P \cdot a_Q = a_Q \cdot a_P \in M \cap L$ *is strictly positive, and we have*

$$\varepsilon_{P\cap Q,P}(C_P^+) \cap \varepsilon_{P\cap Q,Q}(C_Q^+) = C_{P\cap Q}^+. \tag{5.3.2}$$

Proof. (i) The argument is the same as in the proof of Proposition 4.9. Notice that $(a_Q)_{K_Q} = (K_Q a_Q)$ and hence $(a_Q)_{K_P} = (K_P a_Q)$.

Claim. *Let* $X \in H_R(K_P,P)$. *Then* X *centralizes* $(a_Q)_{K_P}$ *if and only if* $X = \sum_{i=1}^r a_i \cdot (K_P \ell_i)$ *for some* $\ell_i \in L$, $a_i \in R$.

Proof of the claim. Assume that X centralizes $(a_Q)_{K_P}$. Write $X = \sum_{i=1}^r a_i \cdot (K_P g_i)$ for some $g_i \in P$. Since $Q = U_Q L$ we can write $g_i = g_{i,U_Q} g_{i,L}$ with $g_{i,L} \in L$, $g_{i,U_Q} \in U_Q$ for $1 \le i \le r$. As a_Q is strictly positive, there exists $n \in \mathbb{N}$ with $a_Q^n g_{i,U_Q} a_Q^{-n} \in K_{U_Q} \subseteq K_{U_P}$ for all $1 \le i \le r$. Using that a_Q centralizes L we compute

$$\begin{aligned}X \cdot a_Q^n = X \cdot (a_Q)_{K_P}^n = (a_Q)_{K_P}^n \cdot X &= \sum_{i=1}^r a_i \cdot (K_P a_Q^n g_{i,U_Q} g_{i,L})\\ &= \sum_{i=1}^r a_i \cdot (K_P a_Q^n g_{i,U_Q} a_Q^{-n} \cdot g_{i,L} a_Q^n) = \left(\sum_{i=1}^r a_i \cdot (K_P g_{i,L})\right) \cdot a_Q^n\end{aligned}$$

and hence $X = \sum_{i=1}^r a_i \cdot (K_P g_{i,L})$. The converse direction is clear, because $(a_Q)_{K_P} = (K_P a_Q)$ and a_Q centralizes L. □

If $X \in H_P(K_P,P)$ centralizes $(a_Q)_{K_P}$, we can write $X = \sum_{i=1}^r a_i \cdot (K_P \ell_i)$ for some $\ell_i \in L$ by the claim. Hence, $X = \varepsilon_{P,Q}(Y)$ with $Y \coloneqq \sum_{i=1}^r a_i \cdot (K_Q \ell_i) \in C_Q^+$ (using the claim for $P = Q$). Hence, $C_{H_R(K_P,P)}((a_Q)_{K_P}) \subseteq \varepsilon_{P,Q}(C_Q^+)$. The converse inclusion follows similarly.

(ii) Notice that $\mathbf{P} \cap \mathbf{Q}$ is a parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{M} \cap \mathbf{L}$. As a_P centralizes $T \subseteq M$ we have $a_P \in Z \subseteq M \cap L$. Similarly, we have $a_Q \in Z \subseteq M \cap L$. Since both a_P and a_Q centralize $M \cap L$ they commute with each other and $a_{P\cap Q} \coloneqq a_P a_Q$ lies in the center of $L \cap M$. Recall from Proposition 1.62, (iii) that

$$\begin{aligned}&\langle \alpha, \nu(a_P)\rangle = 0 \quad \text{for } \alpha \in \Sigma_M, && \langle \alpha, \nu(a_Q)\rangle = 0 \quad \text{for } \alpha \in \Sigma_L,\\ &\langle \alpha, \nu(a_P)\rangle < 0 \quad \text{for } \alpha \in \Sigma^+ \setminus \Sigma_M, && \langle \alpha, \nu(a_Q)\rangle < 0 \quad \text{for } \alpha \in \Sigma^+ \setminus \Sigma_L.\end{aligned}$$

Hence,

$$\langle \alpha, \nu(a_{P\cap Q})\rangle = \begin{cases}\langle \alpha, \nu(a_P)\rangle + \langle \alpha, \nu(a_Q)\rangle = 0, & \text{for } \alpha \in \Sigma_M \cap \Sigma_L = \Sigma_{M\cap L};\\ \langle \alpha, \nu(a_P)\rangle + \langle \alpha, \nu(a_Q)\rangle < 0, & \text{for } \alpha \in \Sigma^+ \setminus \Sigma_{M\cap L}.\end{cases}$$

Therefore, $a_{P\cap Q}$ is strictly $(M\cap L)$-positive by Proposition 1.62, (iii).

An application of the claim in (i) shows that $C^+_{P\cap Q}$ centralizes both $(a_P)_{K_{P\cap Q}}$ and $(a_Q)_{K_{P\cap Q}}$, and hence is contained in $\varepsilon_{P\cap Q,P}(C^+_P)\cap\varepsilon_{P\cap Q,Q}(C^+_Q)$.

Conversely, the elements in $\varepsilon_{P\cap Q,P}(C^+_P)\cap\varepsilon_{P\cap Q,Q}(C^+_Q)$ centralize $(a_P)_{K_{P\cap Q}}$ and $(a_Q)_{K_{P\cap Q}}$, and hence also $(a_{P\cap Q})_{K_{P\cap Q}}=(a_P)_{K_{P\cap Q}}\cdot(a_Q)_{K_{P\cap Q}}$. This proves (5.3.2). □

Corollary 5.20. *Let* $\mathbf{Q}=\mathbf{L}\mathbf{U}_Q$ *be a parabolic subgroup of* $\mathbf{G}$ *with* $\mathbf{P}\subseteq\mathbf{Q}$. *Let* $a_Q\in L^+$ *be a strictly positive element. For any* $X\in H_R(K_P,P)$ *there exists* $n\in\mathbb{N}$ *such that* $(a_Q)^n_{K_P}\cdot X\in\varepsilon_{P,Q}(C^+_Q)$.

Proof. Let $g=g_{U_Q}g_L\in Q$ with $g_{U_Q}\in U_Q$ and $g_L\in L$. As a_Q is strictly positive there exists $n\in\mathbb{N}$ such that $a_Q^n g_{U_Q}a_Q^{-n}\in K_{U_Q}\subseteq K_{U_P}$, and hence $a_Q^n g\in K_PL$.

Given $X=\sum_{i=1}^r a_i\cdot(K_Pg_i)\in H_R(K_P,P)$, we may thus choose $n\in\mathbb{N}$ large enough such that $a_Q^n g_i\in K_PL$ for all $1\le i\le n$. Then $(a_Q)^n_{K_P}\cdot X\in\varepsilon_{P,Q}(C^+_Q)$ by Proposition 5.19, (i) and the claim in its proof. □

Remark 5.21. Consider the involution $\zeta_P\colon H_R(K_P,P)\to H_R(K_P,P)$, $(g)_{K_P}\mapsto(g^{-1})_{K_P}$ (Proposition 2.10).

(a) If $a_P\in M$ is a strictly positive element, then a_P^{-1} is strictly negative, and we have

$$\zeta_P(C^+_P)=C^-_P:=\left\{X\in H_R(K_P,P)\,\middle|\,X\cdot(a_P^{-1})_{K_P}=(a_P^{-1})_{K_P}\cdot X\right\}. \qquad (5.3.3)$$

Given another parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ with $\mathbf{Q}\subseteq\mathbf{P}$, we have $\zeta_Q\circ\varepsilon_{Q,P}=\varepsilon_{Q,P}\circ\zeta_P$ by Lemma 2.11. Therefore, the analogs of Lemma 5.18 and Proposition 5.19 for C^-_P also hold.

(b) It is easy to see that $\Theta^P_M\circ\zeta_P\neq\zeta_M\circ\Theta^P_M$, where ζ_M is the involution on $H_R(K_M,M)$ of Proposition 2.10. Nevertheless, we have

$$\zeta_P(\operatorname{Ker}\Theta^P_M)=\operatorname{Ker}\Theta^P_M. \qquad (5.3.4)$$

To see this, notice that $\operatorname{Ker}\Theta^P_M$ is generated by elements of the form $(g)_{K_P}-\frac{\nu_M(g)\mu_{U_P}(g)}{\mu_{U_P}(g_M)}\cdot(g_M)_{K_P}$ for $g\in P$ (cf. Proposition 4.3). Applying $\Theta^P_M\circ\zeta_P$ yields

$$\nu_M(g^{-1})\mu_{U_P}(g^{-1})\cdot(g_M^{-1})_{K_M}-\frac{\nu_M(g)\mu_{U_P}(g)}{\mu_{U_P}(g_M)}\cdot\mu_{U_P}(g_M^{-1})\cdot(g_M^{-1})_{K_M}.$$

In the notation of Proposition 4.2 it suffices to prove $\delta(g)=\delta(g_M)$, where $\delta(g):=\mu(g)\cdot\mu(g^{-1})^{-1}$ for $g\in P$. By Remark 2.4, (b) the map $\delta\colon P\to\mathbb{Q}^\times$ is a group homomorphism, and satisfies $\delta(g)=[K_P:g^{-1}K_Pg]$. Consequently, we need to show $\delta(g)=1$, whenever $g\in U_P$. But if $g\in U_P$, there exists a compact open subgroup $\Gamma\subseteq P$ with $g\in\Gamma$. Remark 2.4, (c) now implies $\delta(g)=[\Gamma,g^{-1}\Gamma g]=1$. Hence, (5.3.4) holds.

5.4. Decomposition of Hecke polynomials

Assume from now on that p is invertible in R. Let $\mathbf{P} = \mathbf{M}\mathbf{U}_P$ be a parabolic subgroup of $\mathbf{G}$. We view the spherical Hecke algebra $H_R(K, G)$ as a subalgebra of $H_R(K_P, P)$ via the embedding $\varepsilon_{P,G}$, which in turn we view as a subalgebra of $H_R(K_B, B)$ via the embedding $\varepsilon_{B,P}$. Fix a strictly M-positive element $a_P \in Z$ and denote its image in Λ by λ_P. Then $W_{0,M}$ stabilizes $e^{-\lambda_P} \in R[\Lambda]$ with respect to the twisted action. Let W_0^M be a representing system of $W_0/W_{0,M}$ in W_0.

Consider the polynomial

$$\widetilde{\chi}_{a_P}(t) := \prod_{w \in W_0^M} \left(1 - w \star e^{-\lambda_P} \cdot t\right) \in 1 + tR[\Lambda][t].$$

By construction it satisfies $\widetilde{\chi}_{a_P}(e^{\lambda_P}) = 0$, and its coefficients are W_0-invariant under the twisted action. By Theorem 5.10 there exists a unique polynomial

$$\chi_{a_P}(t) = \sum_{i=0}^{|W_0^M|} X_i \cdot t^i \in 1 + tH_R(K, G)[t] \tag{5.4.1}$$

with $\sum_{i=0}^{|W_0^M|} \mathcal{S}(X_i) \cdot t^i = \widetilde{\chi}_{a_P}(t)$. Notice that the constant term X_0 of $\chi_{a_P}(t)$ is 1. By construction we have

$$\mathcal{S}(X_i) = (-1)^i \mu_U(-\lambda_P)^{-i} \cdot \sum_{\substack{\mathcal{R} \subseteq W_0^M, \\ |\mathcal{R}| = i}} \prod_{w \in \mathcal{R}} \mu_U\left(w(-\lambda_P)\right) \cdot e^{\sum_{w \in \mathcal{R}} w(-\lambda_P)} \in R[\Lambda].$$

Lemma 5.22. *We have*

$$\chi_{a_P}\left((a_P)_{K_P}\right) := \sum_{i=0}^{|W_0^M|} (a_P)_{K_P}^i \cdot X_i \in \operatorname{Ker} \Theta_M^P$$

Proof. By Proposition 5.15 we have $\mathcal{S}_M\left(\Theta_M^P\left((a_P)_{K_P}\right)\right) = \Theta_Z^B\left((a_P)_{K_P}\right) = e^{\lambda_P}$ and $\mathcal{S}_M \circ \Theta_M^P\big|_{H_R(K,G)} = \mathcal{S}$. Since, moreover, $\mathcal{S}_M$ is injective by Theorem 5.10, the assertion follows from

$$(\mathcal{S}_M \circ \Theta_M^P)\left(\chi_{a_P}\left((a_P)_{K_P}\right)\right) = \sum_{i=0}^{|W_0^M|} e^{i\lambda_P} \cdot \mathcal{S}(X_i) = \widetilde{\chi}_{a_P}\left(e^{\lambda_P}\right) = 0.$$

□

In fact, we believe that more is true in general.

Conjecture 5.23. *The element $(a_P)_{K_P}$ is a left root of $\chi_{a_P}(t)$, i. e. we have*

$$\chi_{a_P}\left((a_P)_{K_P}\right) = 0 \qquad \text{in } H_R(K_P, P).$$

There is considerable evidence that this conjecture holds true: Andrianov [And77] essentially proved it for $G = \mathrm{Sp}_{2n}(F)$ with P being the "Siegel parabolic", i. e. the subgroup of matrices whose lower left quadrant is zero. Gritsenko subsequently adopted the methods of Andrianov to prove it for $G = \mathrm{GL}_n(F)$ and all parabolics [Gri88, Gri92]. In [Gri90] Gritsenko verified Conjecture 5.23 for the classical groups $\mathrm{Sp}_{2n}(F)$, $\mathrm{SU}_n(F)$, and $\mathrm{SO}_n(F)$, where P is the standard parabolic fixing a line in the standard representation. In section 5.5 we will develop a new method to verify Conjecture 5.23 for the so-called *non-obtuse* parabolics of G.

Definition 5.24. Consider the submodules

$$\begin{aligned}
\mathscr{O}_P^+ &\coloneqq C_P^+.H_R(K,G) \coloneqq \left\{ \sum_{i=1}^r Y_i Z_i \in H_R(K_P,P) \,\middle|\, Y_i \in C_P^+, Z_i \in H_R(K,G) \right\}, \\
\mathscr{O}_P^- &\coloneqq \zeta_P(\mathscr{O}_P^+) = H_R(K,G).C_P^-.
\end{aligned}$$

Assume that Conjecture 5.23 holds. Then, for every $n \in \mathbb{N}$, we may define recursively the "negative powers" of $(a_P)_{K_P}$ as

$$(a_P)_{K_P}^{-n} \coloneqq -\sum_{i=1}^{|W_0^M|} (a_P)_{K_P}^{i-n} \cdot X_i \in \mathscr{O}_P^+.$$

It should be noted that these are not really negative powers of $(a_P)_{K_P}$, whence the quotation marks. In general, for $n > 1$ we even have $\left((a_P)_{K_P}^{-1}\right)^n \neq (a_P)_{K_P}^{-n}$. However, it is clear from the construction that $(a_P)_{K_P} \cdot (a_P)_{K_P}^{-1} = 1$.

Example 5.25. Let us compute some "negative powers" for $G = \mathrm{GL}_2(F)$. The notations are the same as in Example 5.16. Choose the strictly positive element $a_B = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $(a_B)_{K_B} = X_+$ in $H_R(K_B, B)$. The polynomial

$$(1 - X^{-1}t) \cdot (1 - (qY)^{-1}t) = 1 - (X^{-1} + (qY)^{-1}) \cdot t + (qXY)^{-1} \cdot t^2 \in R[X^{\pm 1}, Y^{\pm 1}][t]$$

annihilates $\Theta_T^B\big((a_B)_{K_B}\big) = X$. Under the Satake homomorphism we have

$$\begin{aligned}
\mathcal{S}\left(q^{-1}(\pi E_2)_K^{-1} \cdot (\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_K\right) &= q^{-1}(XY)^{-1}(X + qY) = X^{-1} + (qY)^{-1}, \\
\mathcal{S}\big(q^{-1}(\pi E_2)_K^{-1}\big) &= (qXY)^{-1}.
\end{aligned}$$

Moreover, we compute

$$\begin{aligned}
\varepsilon_{B,G}\left(q^{-1}(\pi E_2)_K^{-1} \cdot (\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix})_K\right) &= q^{-1}Z^{-1} \cdot (X_+ + X_- Z) = q^{-1}X_+ Z^{-1} + q^{-1}X_-, \\
\varepsilon_{B,G}\left(q^{-1}(\pi E_2)_K^{-1}\right) &= q^{-1}Z^{-1}.
\end{aligned}$$

Hence, we have

$$\chi_{a_B}(t) = 1 - q^{-1}(X_+ Z^{-1} + X_-) \cdot t + q^{-1}Z^{-1} \cdot t^2 \in H_R(K,G)[t]. \tag{5.4.2}$$

One readily checks that X_+ is a left root of $\chi_{a_B}(t)$. Hence, the first "negative powers" are

$$\begin{aligned} X_+^{-1} &= q^{-1}(X_+Z^{-1} + X_-) - q^{-1}X_+Z^{-1} = q^{-1}X_-, \\ X_+^{-2} &= q^{-2}X_-(X_+Z^{-1} + X_-) - q^{-1}Z^{-1} = q^{-2}X_-^2 + q^{-2}(X_-X_+ - q \cdot 1)Z^{-1} \\ X_+^{-3} &= q^{-1}X_+^{-2}(X_+Z^{-1} + X_-) - q^{-1}X_+^{-1}Z^{-1} \\ &= q^{-3}X_-^3 + q^{-3}(X_-X_+ - q \cdot 1)X_+Z^{-2} + q^{-3}X_-(X_-X_+ - q \cdot 1)Z^{-1}. \end{aligned}$$

Notice that $X_+^{-2} \neq (X_+^{-1})^2$ and $X_+^{-3} \neq (X_+^{-1})^3$, and that both $X_+^{-2} - (X_+^{-1})^2$ and $X_+^{-3} - (X_+^{-1})^3$ lie in the kernel of Θ_T^B.

Lemma 5.26. *Assume that Conjecture 5.23 holds.*

(i) For every $X \in H_R(K, G)$ and every $n \in \mathbb{N}$ with $(a_P)_{K_P}^n X \in C_P^+$ we have

$$(a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^d = (a_P)_{K_P}^{n+d} \cdot X, \quad \textit{for all } d \in \mathbb{Z}.$$

(ii) For every $X \in \mathcal{O}_P^+$ we have

$$(a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^{-n} = X, \qquad \textit{for } n \gg 0.$$

Proof. (i) Let $X \in H_R(K, G)$ and $n \in \mathbb{N}$ with $(a_P)_{K_P}^n X \in C_P^+$. We do descending induction on d. If $d \geq 0$, then we have

$$\big((a_P)_{K_P}^n X\big) \cdot (a_P)_{K_P}^d = (a_P)_{K_P}^d \cdot \big((a_P)_{K_P}^n X\big) = (a_P)_{K_P}^{n+d} X,$$

by the assumption that $(a_P)_{K_P}^n X$ centralizes $(a_P)_{K_P}$. Now assume $d < 0$. Using that $H_R(K, G)$ is commutative by Theorem 5.10, we compute

$$\begin{aligned} (a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^d &= -\sum_{i=1}^{|W_0^M|} (a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^{i+d} \cdot X_i \\ &= -\sum_{i=1}^{|W_0^M|} (a_P)_{K_P}^{n+i+d} \cdot X \cdot X_i \\ &= \left(-\sum_{i=1}^{|W_0^M|} (a_P)_{K_P}^{n+i+d} \cdot X_i\right) \cdot X \\ &= (a_P)_{K_P}^{n+d} \cdot X, \end{aligned}$$

where we use the induction hypothesis for the second equality.

(ii) Write $X = \sum_{j=1}^r Y_j Z_j$ with $Y_j \in C_P^+$ and $Z_j \in H_R(K, G)$. Choose $n \in \mathbb{N}$ such that $(a_P)_{K_P}^n Z_j \in C_P^+$ for all $1 \leq j \leq r$. By (i) we have

$$(a_P)_{K_P}^n \cdot X \cdot (a_P)_{K_P}^{-n} = \sum_{j=1}^r Y_j \cdot (a_P)_{K_P}^n Z_j (a_P)_{K_P}^{-n} = \sum_{j=1}^r Y_j Z_j = X.$$

□

Proposition 5.27. *The following assertions are equivalent:*

(i) Conjecture 5.23 holds;

(ii) we have $\mathscr{O}_P^+ \cap \operatorname{Ker}\Theta_M^P = \{0\}$, i. e. the restriction of Θ_M^P to $\mathscr{O}_P^+$ is injective;

(iii) we have $\mathscr{O}_P^- \cap \operatorname{Ker}\Theta_M^P = \{0\}$, i. e. the restriction of Θ_M^P to $\mathscr{O}_P^-$ is injective.

Proof. We have $\chi_{a_P}\big((a_P)_{K_P}\big) \in \mathscr{O}_P^+ \cap \operatorname{Ker}\Theta_M^P$, hence (ii) implies (i).

Assume (i). Let $X \in \mathscr{O}_P^+$ with $X \neq 0$. By Corollary 5.20 there exists $n \in \mathbb{N}$ with $(a_P)_{K_P}^n X \in C_P^+$. By Lemma 5.26, (ii) we have $(a_P)_{K_P}^n X \cdot (a_P)_{K_P}^{-n} = X \neq 0$, and hence $(a_P)_{K_P}^n X \neq 0$. As Θ_M^P is injective on C_P^+ by Lemma 5.18, (ii), we deduce

$$\Theta_M^P\big((a_P)_{K_P}^n\big) \cdot \Theta_M^P(X) = \Theta_M^P\big((a_P)_{K_P}^n X\big) \neq 0.$$

This shows $\Theta_M^P(X) \neq 0$ and proves (ii).

The equivalence of (ii) and (iii) now follows from (5.3.4). □

Corollary 5.28. *Assume Conjecture 5.23. We have direct sum decompositions*

$$H_R(K_P, P) = \mathscr{O}_P^+ \oplus \operatorname{Ker}\Theta_M^P = \mathscr{O}_P^- \oplus \operatorname{Ker}\Theta_M^P.$$

Proof. By Proposition 5.27 we have $\mathscr{O}_P^+ \cap \operatorname{Ker}\Theta_M^P = \{0\}$. Given $X \in H_R(K_P, P)$, let $n \in \mathbb{N}$ with $(a_P)_{K_P}^n X \in C_P^+$. Then $(a_P)_{K_P}^n X \cdot (a_P)_{K_P}^{-n} \in \mathscr{O}_P^+$ and $X - (a_P)_{K_P}^n X \cdot (a_P)_{K_P}^{-n} \in \operatorname{Ker}\Theta_M^P$. Whence the first decomposition. The second one follows from the first by applying ζ_P and using (5.3.4). □

Notation 5.29. Given a morphism $\psi\colon A \to B$ of R-algebras and a polynomial $f(t) = \sum_{i=0}^n a_i t^i \in A[t]$, we denote by $f^\psi(t) := \sum_{i=0}^n \psi(a_i) \cdot t^i \in B[t]$ the polynomial obtained from applying ψ to the coefficients of $f(t)$.

We recall the following commutative diagram (Proposition 5.15):

$$\begin{tikzcd}
H_R(K_B, B) \arrow[rrd, bend left, "\Theta_Z^B"] & & \\
H_R(K_P, P) \arrow[u, "\varepsilon_{B,P}"] \arrow[r, "\Theta_M^P"] & H_R(K_M, M) \arrow[r, "\mathcal{S}_M"] & H_R(K_Z, Z) \\
H_R(K, G) \arrow[u, "\varepsilon_{P,G}"] \arrow[rru, bend right, "\mathcal{S}"'] & &
\end{tikzcd}$$

Theorem 5.30. *Assume Conjecture 5.23. Let $d(t) \in H_R(K, G)[t]$ be a polynomial such that $d^{\mathcal{S}}(t)$ decomposes in $R[\Lambda][t]$ as $d^{\mathcal{S}}(t) = \widetilde{f}(t) \cdot \widetilde{g}(t)$, where either $\widetilde{f}(t)$ has coefficients in $(\mathcal{S}_M \circ \Theta_M^P)(C_P^+)$ with constant term 1 or $\widetilde{g}(t)$ has coefficients in $(\mathcal{S}_M \circ \Theta_M^P)(C_P^-)$ with constant term 1.*

Then there exist polynomials $f(t), g(t) \in H_R(K_P, P)[t]$ *with* $\deg f(t) = \deg \widetilde{f}(t)$, $\deg g(t) = \deg \widetilde{g}(t)$, $f^{\mathcal{S}_M \circ \Theta_M^P}(t) = \widetilde{f}(t)$ *and* $g^{\mathcal{S}_M \circ \Theta_M^P}(t) = \widetilde{g}(t)$, *and such that*

$$d(t) = f(t) \cdot g(t) \quad \textit{in } H_R(K_P, P)[t].$$

Proof. The above diagram shows $\Theta_Z^B\big|_{H_R(K_P,P)} = \mathcal{S}_M \circ \Theta_M^P$ and $\Theta_Z^B\big|_{H_R(K,G)} = \mathcal{S}$.

We only prove the case where $f(t) \in \Theta_Z^B(C_P^+)[t]$; the other case is completely analogous. It follows from Proposition 5.27 and Theorem 5.10 that Θ_Z^B is injective on $\mathcal{O}_P^+$. Hence there exists a unique polynomial $f(t) \in C_P^+[t]$ with $f^{\Theta_Z^B}(t) = \widetilde{f}(t)$. Its constant term is necessarily 1, so there exists a power series $e(t) = \sum_{i=0}^{\infty} e_i t^i \in C_P^+[\![t]\!]$ with $e(t) \cdot f(t) = f(t) \cdot e(t) = 1$. We consider the power series

$$g(t) := e(t) \cdot d(t) \in \mathcal{O}_P^+[\![t]\!].$$

We then have

$$g^{\Theta_Z^B}(t) = e^{\Theta_Z^B}(t) \cdot d^{\Theta_Z^B}(t) = e^{\Theta_Z^B}(t) \cdot \widetilde{f}(t) \cdot \widetilde{g}(t) = \widetilde{g}(t).$$

As Θ_Z^B is injective on $\mathcal{O}_P^+$, it follows that $g(t)$ is indeed a polynomial of degree $\deg \widetilde{g}(t)$, and we have $f(t) \cdot g(t) = f(t) \cdot e(t) \cdot d(t) = d(t)$. □

Remark 5.31. (a) With the notations and assumptions of Theorem 5.30 the polynomial $d^{\mathcal{S}}(t)$ already decomposes over $\mathcal{S}_M\big(H_R(K_M, M)\big)$.

(b) Assume Conjecture 5.23. Let $\mathbf{Q}$ be another parabolic subgroup of $\mathbf{G}$ with $\mathbf{Q} \subseteq \mathbf{P}$. Proposition 5.19, (ii) implies $C_Q^+ \subseteq C_P^+$, and hence $\mathcal{O}_Q^+ \subseteq \mathcal{O}_P^+$. By Proposition 5.27 it follows that Conjecture 5.23, and hence also Theorem 5.30, holds if we replace $\mathbf{P}$ by $\mathbf{Q}$.

Consequently, it suffices to prove Conjecture 5.23 for *maximal* parabolic subgroups.

We draw some consequences of Theorem 5.30, cf. [And77, Thm. 6.2 and Cor.].

Corollary 5.32. *Assume Conjecture 5.23.*

(i) Let $f(t) \in C_P^+[t]$ *be a polynomial with constant term* 1. *Then there exists a polynomial* $g(t) \in H_R(K_P, P)[t]$ *with constant term* 1 *and* $\deg g(t) \le \deg f(t) \cdot \big(|W_0^M| - 1\big)$ *such that*

$$d(t) := f(t) \cdot g(t) \in H_R(K, G)[t].$$

(ii) Let $g(t) \in C_P^-[t]$ *be a polynomial with constant term* 1. *Then there exists a polynomial* $f(t) \in H_R(K_P, P)[t]$ *with constant term* 1 *and* $\deg f(t) \le \deg g(t) \cdot \big(|W_0^M|\big)$ *such that*

$$d(t) := f(t) \cdot g(t) \in H_R(K, G)[t].$$

(iii) Let $X \in C_P^+$. *There exists a monic polynomial* $d(t) = \sum_{i=0}^{r} d_i t^i \in H_R(K, G)[t]$ *such that* $\deg d(t) \le |W_0^M|$ *and*

$$\sum_{i=0}^{r} X^i d_i = 0.$$

(iv) *Let* $X \in C_P^-$. *There exists a monic polynomial* $d(t) = \sum_{i=0}^r d_i t^i \in H_R(K,G)[t]$ *such that* $\deg d(t) \leq |W_0^M|$ *and*

$$\sum_{i=0}^r d_i X^i = 0.$$

Proof. Notice that (ii) and (iv) are completely analogous to (i) and (iii), respectively. We will therefore only address (i) and (iii).

(i) Write $f^{\Theta_Z^B}(t) =: \widetilde{f}(t) = \sum_{i=0}^r f_i t^i \in R[\Lambda][t]$. By assumption the coefficients of $\widetilde{f}(t)$ lie in $\mathcal{S}_M(H_R(K_M, M))$, hence they are invariant under the twisted action of $W_{0,M}$. Given $w \in W_0$, write $\widetilde{f}^w(t) = \sum_{i=0}^r w \star f_i \cdot t^i$. The polynomial $\widetilde{d}(t) := \prod_{w \in W_0^M} \widetilde{f}^w(t)$ is W_0-invariant with respect to the twisted action, and hence has coefficients in $\mathcal{S}(H_R(K,G))$. Moreover, it factors as $\widetilde{d}(t) = \widetilde{f}(t) \cdot \widetilde{g}(t)$ for some $\widetilde{g}(t) \in R[\Lambda][t]$ with constant term 1 and $\deg \widetilde{g} \leq r \cdot \big(|W_0^M| - 1\big)$. The assertion now follows from Theorem 5.30.

(iii) Let $X \in C_P^+$. Applying (i) to $f(t) := 1 - Xt$ yields a polynomial $g(t) = \sum_{i=0}^{r-1} g_i t^i \in H_R(K_P, P)[t]$, where $g_0 = 1$ and $r \leq |W_0^M|$, such that

$$f(t)g(t) =: \sum_{i=0}^r d_{r-i} t^i \in H_R(K,G)[t].$$

Comparing coefficients yields $d_r = 1$, $d_0 = -Xg_{r-1}$, and $d_i = g_{r-i} - Xg_{r-i-1}$ for $1 \leq i \leq r-1$. Hence,

$$\sum_{i=0}^r X^i d_i = -Xg_{r-1} + \sum_{i=1}^{r-1} \big(X^i g_{r-i} - X^{i+1} g_{r-(i+1)}\big) + X^r = 0. \qquad \square$$

Example 5.33. We continue Example 5.25, where $G = \mathrm{GL}_2(F)$. Recall the notation $X_+ = (a_B)_{K_B} = (\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right))_{K_B}$, $X_- = (\left(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right))_{K_B}$, and $Z = (\pi E_2)_{K_B}$ in the parabolic Hecke algebra $H_R(K_B, B)$. Recall also the relation $X_+ X_- = q \cdot 1$ in $H_R(K_B, B)$. Consider the polynomial (5.4.2)

$$\chi_{a_B}(t) = 1 - q^{-1}(X_+ Z^{-1} + X_-) \cdot t + q^{-1} Z^{-1} \cdot t^2.$$

We will compute its decomposition according to Theorem 5.30.

The image of $\chi_{a_B}(t)$ in $R[X^{\pm 1}, Y^{\pm 1}][t]$ decomposes as $f(t) \cdot g(t)$ with $f(t) := 1 - (qY)^{-1} \cdot t$ and $g(t) := 1 - X^{-1} t$. The polynomial $F(t) := 1 - q^{-1} X_+ Z^{-1} \cdot t$ has coefficients in C_B^+ and maps to $f(t)$ under Θ_T^B. If we let $E(t) = \sum_{k=0}^\infty (q^{-1} X_+ Z^{-1})^k t^k$ be the inverse of $F(t)$ in $C_B^+[\![t]\!]$, then we compute

$$\begin{aligned}
G(t) &= E(t) \cdot \chi_{a_B}(t) \\
&= \underbrace{E(t) - E(t) \cdot (q^{-1} X_+ Z^{-1}) t}_{=1} - E(t) \cdot q^{-1} X_- t + E(t) \cdot q^{-1} Z^{-1} t^2 \\
&= 1 - q^{-1} X_- \cdot t - \sum_{k=1}^\infty q^{-k} X_+^{k-1} Z^{-k} t^{k+1} + \sum_{k=0}^\infty q^{-k-1} X_+^k Z^{-k-1} t^{k+2} \\
&= 1 - q^{-1} X_- \cdot t.
\end{aligned}$$

Hence, $\chi_{a_B}(t)$ decomposes in $H_R(K_B, B)[t]$ as

$$\chi_{a_B}(t) = (1 - q^{-1}X_+ Z^{-1} \cdot t) \cdot (1 - q^{-1}X_- \cdot t).$$

5.5. Verification of Conjecture 5.23 for special cases

Let $\mathbf{P} = \mathbf{M}\mathbf{U}_P$ be a maximal parabolic subgroup of $\mathbf{G}$. Recall the fixed W_0-invariant scalar product $(\cdot, \cdot)$ on V^* (cf. section 1.1). In this section we will also view the special point φ_0 of $\mathcal{A}$ as a valuation on the root group datum $(Z, (U_\alpha)_{\alpha\in\Phi})$ of G.

Definition 5.34. We call $\mathbf{P}$ *non-obtuse* if

$$(\alpha, \beta) \geq 0, \qquad \text{for all } \alpha, \beta \in \Sigma^+ \smallsetminus \Sigma_M. \tag{5.5.1}$$

We assume from now on that $\mathbf{P}$ is non-obtuse. Under this condition we will prove Conjecture 5.23.

Fix a representing system W_0^M of $W_0/W_{0,M}$ in W_0. Recall from section 5.4 the polynomial $\chi_{a_P}(t) = \sum_{i=0}^{|W_0^M|} X_i \cdot t^i$, where the $X_i \in H_R(K, G)$ are the unique elements with

$$\mathcal{S}(X_i) = (-1)^i \mu_U(-\lambda_P)^{-i} \cdot \sum_{\substack{\mathcal{R}\subseteq W_0^M \\ |\mathcal{R}|=i}} \prod_{w\in\mathcal{R}} \mu_U\big(w(-\lambda_P)\big) \cdot e^{\sum_{w\in\mathcal{R}} w(-\lambda_P)} \in R[\Lambda]; \tag{5.5.2}$$

we mention that $\sum_{w\in\mathcal{R}} \nu\big(w(-\lambda_P)\big) \leq \nu(-i\lambda_P)$ [Bou81, Ch. VI, §1.6, Prop. 18], where $|\mathcal{R}| = i$.

It suffices to prove $(a_P)^i_{K_P} X_i \in C_P^+$ for all $i = 0, \dots, |W_0^M|$, for then it will follow from Lemmas 5.22 and 5.18 that

$$\chi_{a_P}\big((a_P)_{K_P}\big) \in C_P^+ \cap \operatorname{Ker}\Theta_M^P = \{0\}.$$

As K_M normalizes U_P it follows from the Iwasawa decomposition 5.7, (ii) applied to G and the Cartan decomposition 5.7, (i) applied to M that every coset in $K\backslash G$ is of the form $Kuzk$ with $u \in U_P$, $z \in Z^{+,M}$ and $k \in K_M$. Thus, we may write $X_i = \sum_{j=1}^{r_i} c_{i,j} \cdot (K_P u_{i,j} z_{i,j} k_{i,j})$ in $H_R(K_P, P)$ with $u_{i,j} \in U_P$, $z_{i,j} \in Z^{+,M}$ and $k_{i,j} \in K_M$. It follows from (5.5.2) and Remarks 5.11, 5.8 that $\nu(z_{i,j}) \leq \nu(-i\lambda_P)$ for all i, j. In view of the claim in the proof of Proposition 5.19 (for $\mathbf{Q} = \mathbf{P}$) it suffices to prove $a_P^i u_{i,j} a_P^{-i} \in K_P$ for all $j = 1, \dots, r_i$, for then we have

$$(a_P)^i_{K_P} \cdot X_i = \sum_{j=1}^{r_i} c_{i,j} \cdot (K_P a_P^i u_{i,j} a_P^{-i} \cdot a_P^i z_{i,j} k_{i,j}) \in C_P^+.$$

Hence, the truth of the statement in Conjecture 5.23 for non-obtuse parabolics is a consequence of the following theorem.

Theorem 5.35. *Let $a \in Z$ be strictly M-positive. Let $u \in U_P$, $z \in Z^-$ and $z' \in Z$ satisfying $\nu(z) \leq \nu(a^{-1})$ and $uz' \in KzK$. Then we have*

(i) $az' \in M^+$;
(ii) $aua^{-1} \in K_P$.

In the rest of this section we will prove the first half of Theorem 5.35, give the classification of the non-obtuse parabolics, and finally prove the second half of Theorem 5.35.

Remark 5.36. If $uz' \in KzK$ for some $u \in U$, $z' \in Z$, and $z \in Z^-$, then we have $w.\nu(z') \leq \nu(z)$ for all $w \in W_0$. This holds since for every $w \in W_0$ we find some $u_w \in U$ with $u_w(z')^{n_w^{-1}} \in KzK$, as $\mathcal{S}((z)_K)$ is W_0-invariant. (Here, $n_w \in N$ denotes a lift of w.)

Therefore, Theorem 5.35, (i) follows from the next lemma.

Lemma 5.37. *Let $\lambda \in \Lambda$ be strictly positive, and let $\mu \in \Lambda$ with $\nu(w(\mu)) \leq \nu(-\lambda)$ for all $w \in W_0$. Then we have $\lambda + \mu \in \Lambda_{M^+}$, i. e. $\langle \alpha, \nu(\lambda + \mu)\rangle \leq 0$ for all $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$.*

Proof. Notice that since $\mathbf{P}$ is non-obtuse, we have $\langle \alpha, \beta^\vee \rangle \geq 0$ for all $\alpha, \beta \in \Sigma^+ \smallsetminus \Sigma_M$.

We first assume $\mu = w(-\lambda)$ for some $w \in W_0$ and prove the statement by induction on $\ell(w)$. If $\ell(w) = 1$, write $w = s_\beta$ for some $\beta \in \Delta$. For each $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$ we have

$$\langle \alpha, \nu(\lambda + s_\beta(-\lambda))\rangle = \langle \beta, \nu(\lambda)\rangle \cdot \langle \alpha, \beta^\vee \rangle \leq 0.$$

Now, assume $\ell(w) > 1$. Take $\beta \in \Delta$ with $\ell(s_\beta w) < \ell(w)$ and write

$$\nu\big(\lambda + w(-\lambda)\big) = \nu\big(\lambda + s_\beta(-\lambda)\big) + s_\beta \nu\big(\lambda + s_\beta w(-\lambda)\big).$$

If $\beta \in \Sigma_M$, then we have $s_\beta(\Sigma^+ \smallsetminus \Sigma_M) = \Sigma^+ \smallsetminus \Sigma_M$, and hence the statement follows from the base case and from

$$\big\langle \alpha, s_\beta \nu\big(\lambda + s_\beta w(-\lambda)\big)\big\rangle = \big\langle s_\beta(\alpha), \nu\big(\lambda + s_\beta w(-\lambda)\big)\big\rangle \leq 0,$$

where we have used the induction hypothesis.

If however $\beta \in \Sigma^+ \smallsetminus \Sigma_M$, we compute for each $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$

$$\begin{aligned}
\big\langle \alpha, \nu\big(\lambda + w(-\lambda)\big)\big\rangle &= \big\langle \alpha, \nu\big(\lambda + s_\beta(-\lambda)\big)\big\rangle + \big\langle \alpha, s_\beta \nu\big(\lambda + s_\beta w(-\lambda)\big)\big\rangle \\
&= \langle \beta, \nu(\lambda)\rangle \cdot \langle \alpha, \beta^\vee \rangle + \big\langle \alpha - \langle \alpha, \beta^\vee\rangle \cdot \beta, \nu\big(\lambda + s_\beta w(-\lambda)\big)\big\rangle \\
&= \big\langle \alpha, \nu\big(\lambda + s_\beta w(-\lambda)\big)\big\rangle - \langle \alpha, \beta^\vee \rangle \cdot \big\langle \beta, \nu\big(s_\beta w(-\lambda)\big)\big\rangle \\
&= \big\langle \alpha, \nu\big(\lambda + s_\beta w(-\lambda)\big)\big\rangle + \langle \alpha, \beta^\vee \rangle \cdot \big\langle (s_\beta w)^{-1}(\beta), \nu(\lambda)\big\rangle \\
&\leq 0,
\end{aligned}$$

where for the inequality we have used the induction hypothesis and $(s_\beta w)^{-1}(\beta) \in \Sigma^+$, which in turn follows from $\ell\big((s_\beta w)^{-1} s_\beta\big) > \ell\big((s_\beta w)^{-1}\big)$. This finishes the induction step.

Now, let μ be general. Take $\alpha \in \Sigma^+ \smallsetminus \Sigma_M$. Let $\alpha_0 = w(\alpha)$ be a root of maximal height in the W_0-orbit of α. Then we have $\langle \alpha_0, \beta^\vee \rangle \geq 0$ for all $\beta \in \Sigma^+$, for otherwise

$s_\beta(\alpha_0) = \alpha_0 - \langle \alpha_0, \beta^\vee \rangle \cdot \beta$ would have greater height than α_0. By the hypothesis $\nu(-\lambda) - \nu(w(\mu))$ is a sum of positive coroots, and hence

$$\left\langle \alpha, \nu\left(w^{-1}(\lambda) + \mu\right)\right\rangle = \left\langle w(\alpha), \nu\left(\lambda + w(\mu)\right)\right\rangle = -\left\langle \alpha_0, \nu(-\lambda) - \nu\left(w(\mu)\right)\right\rangle \leq 0.$$

Together with the first part, we obtain

$$\langle \alpha, \nu(\lambda + \mu)\rangle = \langle \alpha, \nu(\lambda + w^{-1}(-\lambda))\rangle + \langle \alpha, \nu(w^{-1}(\lambda) + \mu)\rangle \leq 0. \qquad \square$$

Since $\mathbf{P}$ is assumed to be a *maximal* parabolic, it follows that $\Sigma^+ \smallsetminus \Sigma_M$ is contained in some irreducible component of Σ. Without loss of generality we may thus assume that Σ itself is irreducible. If Δ' is any basis of Σ we denote by $\Sigma^+_{\Delta'}$ the positive roots with respect to Δ'.

Lemma 5.38. *Let $s_1 \cdots s_r$ be a reduced decomposition of the longest element w_0 in W_0, where $s_i := s_{\alpha_i}$ for $\alpha_i \in \Delta$. Put $\beta_i := s_1 \cdots s_{i-1}(\alpha_i)$ for $i = 1, \ldots, r$. Then we have*

$$\Sigma^+_{s_1 \cdots s_j(\Delta)} = \{\beta_{j+1}, \beta_{j+2}, \ldots, \beta_r, -\beta_1, -\beta_2, \ldots, -\beta_j\} \quad \textit{for all } 0 \leq j \leq r. \tag{5.5.3}$$

Proof. Confer [Mac71, Lem. (2.1.1)]. Write $w_j = s_1 \cdots s_j$. Applying [Bou81, Ch. VI, §1.6, Cor. 2 of Prop. 17] to $w_j^{-1} = s_j \cdots s_1$ yields

$$\Sigma^+_\Delta \cap \Sigma^-_{w_j(\Delta)} = \Sigma^+_\Delta \cap w_j \Sigma^-_\Delta = \{\beta_1, \beta_2, \ldots, \beta_j\}.$$

Now, the assertion follows from $\Sigma^+_\Delta = \Sigma^+_\Delta \cap \Sigma^-_{w_0(\Delta)} = \{\beta_1, \beta_2, \ldots, \beta_r\}$ and

$$\Sigma^+_{w_j(\Delta)} = (\Sigma^-_\Delta \cap \Sigma^+_{w_j(\Delta)}) \sqcup (\Sigma^+_\Delta \cap \Sigma^+_{w_j(\Delta)}) = -(\Sigma^+_\Delta \cap \Sigma^-_{w_j(\Delta)}) \sqcup (\Sigma^+_\Delta \smallsetminus \Sigma^-_{w_j(\Delta)}). \qquad \square$$

The maximal parabolic subgroups of $\mathbf{G}$ are in one-to-one correspondence with the elements of Δ (cf. section 1.8). We write $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and let $\mathbf{P}_i = \mathbf{M}_i \mathbf{U}_{P_i}$, for $1 \leq i \leq n$, be the maximal parabolic with root system $\Sigma_{P_i} = \Sigma^+ \cup \Sigma_{\Delta \smallsetminus \{\alpha_i\}}$. We put $\Sigma_{U_{P_i}} := \Sigma^+ \smallsetminus \Sigma_{M_i}$. Then α_i is the unique element in $\Sigma_{U_{P_i}} \cap \Delta$. We call α_i *non-obtuse* if $\mathbf{P}_i$ has this property.

Proposition 5.39. *(i) The classification of non-obtuse simple roots is given in terms of the Dynkin diagram of Σ in Figure 3 below. In types E_8, F_4 and G_2 there are no non-obtuse roots.*

(ii) Let $\alpha \in \Delta \cap \Sigma_{U_P}$ be the simple root corresponding to $\mathbf{P}$*. The following are equivalent:*

(a) the root α is non-obtuse;

(b) we have $(\alpha, \beta) \geq 0$ for all $\beta \in \Sigma_{U_P}$;

(c) the Weyl group $W_{0,M}$ acts transitively on the roots in Σ_{U_P} of the same length.

Proof. It is clear that the property of being non-obtuse depends only on the reduced roots. We consider each type separately. For the descriptions of the root systems we refer to [Bou81, Pl. I–IX]. As usual, we denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$, endowed with the canonical inner product $(\cdot, \cdot)$. We write $s_i := s_{\alpha_i}$ for all i. Given a root $\beta \in \Sigma$, we write $\beta = \sum_{i=1}^n c_i(\beta)\alpha_i$. We observe $c_i(s_j(\beta)) = c_i(\beta)$ for $i \neq j$, and $c_i(s_i(\beta)) = -c_i(\beta) - \sum_{j \neq i} c_j(\beta) \cdot \langle \alpha_j, \alpha_i^\vee \rangle$. In particular, the action of W_{0,M_i} does *not* affect $c_i(\beta)$.

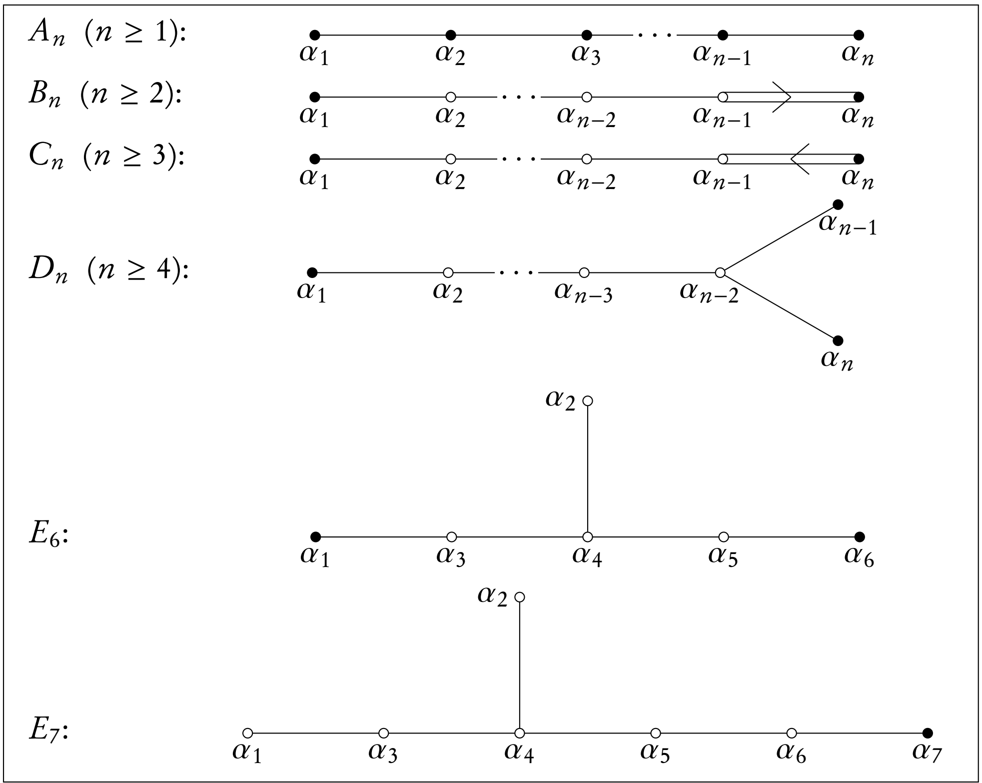

Figure 3: Classification of non-obtuse simple roots. Black vertices denote non-obtuse roots.

(A) Inside $V = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} x_i = 0\right\}$ we consider the root system of type A_n

$$\Sigma = \left\{\pm(e_i - e_j) \,\middle|\, 1 \leq i < j \leq n+1\right\}.$$

A basis is given by $\alpha_i \coloneqq e_i - e_{i+1}$ for $i = 1, \ldots, n$. The roots $e_i - e_j = \sum_{k=i}^{j-1} \alpha_k$, for $1 \leq i < j \leq n+1$, are positive. The finite Weyl group W_0 is the symmetric group acting on $e_1, \ldots, e_{n+1}$. Let $1 \leq r \leq n$. Then we have

$$\Sigma_{U_{P_r}} = \left\{e_i - e_j \,\middle|\, 1 \leq i \leq r < j \leq n+1\right\}.$$

Given $e_i - e_j, e_a - e_b \in \Sigma_{U_{P_r}}$, we observe $a \neq j$ and $i \neq b$, and hence $(e_i - e_j, e_a - e_b) = \delta_{ia} + \delta_{jb} \geq 0$. Therefore, α_r is obtuse. The Weyl group W_{0,M_r} is isomorphic to the product of two symmetric groups acting on $\{e_1, \ldots, e_r\}$ and $\{e_{r+1}, \ldots, e_{n+1}\}$, respectively. It clearly acts transitively on $\Sigma_{U_{P_r}}$.

(B) Inside $V = \mathbb{R}^n$ we consider the root system of type B_n

$$\Sigma = \{\pm e_i \,|\, 1 \leq i \leq n\} \cup \left\{\pm e_i \pm e_j \,\middle|\, 1 \leq i < j \leq n\right\}.$$

A basis is given by $\alpha_i = e_i - e_{i+1}$, for $1 \leq i < n$, and $\alpha_n = e_n$. The positive roots are

$$\begin{cases} e_i = \sum_{k=i}^{n} \alpha_k, & \text{for } 1 \le i \le n; \\ e_i - e_j = \sum_{k=i}^{j-1} \alpha_k, & \text{for } 1 \le i < j \le n; \\ e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{n} \alpha_k, & \text{for } 1 \le i < j \le n. \end{cases}$$

The finite Weyl group W_0 is isomorphic to $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ with $\mathfrak{S}_n$ permuting the e_i, and $(\mathbb{Z}/2\mathbb{Z})^n$ acting by changing signs of the e_i.

Let $r = 1$. We have

$$\Sigma_{U_{P_1}} = \{e_1\} \cup \{e_1 \pm e_i \mid 2 \le i \le n\}.$$

Given $e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j \in \Sigma_{U_{P_1}}$ with $\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$, we compute $(e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j) = 1 + \varepsilon_1\varepsilon_2\delta_{i,j} \ge 0$. Hence, α_1 is non-obtuse. The Weyl group W_{0,M_1} is isomorphic to $\mathfrak{S}_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ with both groups acting on $e_2, \ldots, e_n$, leaving e_1 fixed. It clearly acts transitively on $\{e_1 \pm e_i \mid 2 \le i \le n\}$ (and on $\{e_1\}$).

Let $r = n$. We have

$$\Sigma_{U_{P_n}} = \{e_i \mid 1 \le i \le n\} \cup \left\{e_i + e_j \,\middle|\, 1 \le i < j \le n\right\}.$$

It is obvious that α_n is non-obtuse. The Weyl group W_{0,M_n} is isomorphic to $\mathfrak{S}_n$ acting on $e_1, \ldots, e_n$. It clearly acts transitively on both $\{e_i \mid 1 \le i \le n\}$ and $\left\{e_i + e_j \,\middle|\, 1 \le i < j \le n\right\}$.

Let $1 < r < n$; in particular $n \ge 3$. Notice that both $e_r - e_{r+1}$ and $e_{r-1} + e_{r+1}$ lie in $\Sigma_{U_{P_r}}$ and satisfy $(e_r - e_{r+1}, e_{r-1} + e_{r+1}) = -1$. Hence, α_r is not non-obtuse. The highest root is $\alpha_0 = e_1 + e_2 = \alpha_1 + 2\sum_{k=2}^{n} \alpha_k$. Notice that α_0 and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\alpha_0) = 2 \ne 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof α_0 does not lie in the W_{0,M_r}-orbit of α_r.

(C) Inside $V = \mathbb{R}^n$ we consider the root system of type C_n

$$\Sigma = \{\pm 2e_i \mid 1 \le i \le n\} \cup \left\{\pm e_i \pm e_j \,\middle|\, 1 \le i < j \le n\right\}.$$

A basis is given by $\alpha_i = e_i - e_{i+1}$, for $1 \le i < n$, and $\alpha_n = 2e_n$. The positive roots are

$$\begin{cases} e_i - e_j = \sum_{k=i}^{j-1} \alpha_k, & \text{for } 1 \le i < j \le n; \\ e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{n-1} \alpha_k + \alpha_n, & \text{for } 1 \le i < j \le n; \\ 2e_i = 2\sum_{k=i}^{n-1} \alpha_k + \alpha_n, & \text{for } 1 \le i \le n. \end{cases}$$

The finite Weyl group W_0 is isomorphic to $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ as in the case of type B_n.

Let $r = 1$. We have

$$\Sigma_{U_{P_1}} = \{2e_1\} \cup \{e_1 \pm e_i \mid 2 \le i \le n\}.$$

Given $e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j \in \Sigma_{U_{P_1}}$ with $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $i \ne 1 \ne j$, we have $(e_1 + \varepsilon_1 e_i, e_1 + \varepsilon_2 e_j) = 1 + \varepsilon_1\varepsilon_2\delta_{i,j} \ge 0$. Since clearly $(2e_1, 2e_1) = 4 \ge 0$ and $(2e_1, e_1 \pm e_i) = 2 \ge 0$ for $2 \le i \le n$, the root α_1 is non-obtuse. The Weyl group

W_{0,M_1} is isomorphic to $\mathfrak{S}_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ with both groups acting on $e_2, \ldots, e_n$, leaving e_1 fixed. It clearly acts transitively on $\{e_1 \pm e_i \mid 2 \leq i \leq n\}$ (and on $\{2e_1\}$).

Let $r = n$. We have

$$\Sigma_{U_{P_n}} = \{2e_i \mid 1 \leq i \leq n\} \cup \left\{e_i + e_j \,\middle|\, 1 \leq i < j \leq n\right\}.$$

It is obvious that α_n is non-obtuse. The Weyl group W_{0,M_n} is isomorphic to $\mathfrak{S}_n$ acting on $e_1, \ldots, e_n$. It clearly acts transitively on both $\{2e_i \mid 1 \leq i \leq n\}$ and $\left\{e_i + e_j \,\middle|\, 1 \leq i < j \leq n\right\}$.

Let $1 < r < n$; in particular $n \geq 3$. Notice that both $e_r - e_{r+1}$ and $e_{r-1} + e_{r+1}$ lie in $\Sigma_{U_{P_r}}$ and satisfy $(e_r - e_{r+1}, e_{r-1} + e_{r+1}) = -1$. Hence, α_r is not non-obtuse. Consider the root $\beta = e_1 + e_2 = \alpha_1 + 2\sum_{k=2}^{n-1} \alpha_k + \alpha_n$. Then β and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\beta) = 2 \neq 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof β does not lie in the W_{0,M_r}-orbit of α_r.

(D) Inside $V = \mathbb{R}^n$ we consider the root system of type D_n

$$\Sigma = \left\{\pm e_i \pm e_j \,\middle|\, 1 \leq i < j \leq n\right\}.$$

A basis is given by $\alpha_i = e_i - e_{i+1}$, for $1 \leq i < n$, and $\alpha_n = e_{n-1} + e_n$. The positive roots are

$$\begin{cases} e_i - e_j = \sum_{k=i}^{j-1} \alpha_k, & \text{for } 1 \leq i < j \leq n; \\ e_i + e_n = \sum_{k=i}^{n-2} \alpha_k + \alpha_n, & \text{for } 1 \leq i < n; \\ e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n, & \text{for } 1 \leq i < j < n. \end{cases}$$

The finite Weyl group W_0 is isomorphic to $\mathfrak{S}_n \ltimes \Gamma$, where Γ is the kernel of the map $(\mathbb{Z}/2\mathbb{Z})^n \longrightarrow \mathbb{Z}/2\mathbb{Z}$, $(x_i)_i \mapsto \sum_{i=1}^n x_i$.

Let $r = 1$. We have

$$\Sigma_{U_{P_1}} = \{e_1 \pm e_i \mid 2 \leq i \leq n\}.$$

The same computation as in (B) shows that α_1 is non-obtuse. The Weyl group W_{0,M_1} is isomorphic to $\mathfrak{S}_{n-1} \ltimes \Gamma_1$, where $\mathfrak{S}_{n-1}$ permutes $e_2, \ldots, e_n$ and $\Gamma_1 \subseteq \Gamma$ is the subgroup of elements $(x_i)_i$ with $x_1 = 0$. W_{0,M_1} clearly acts transitively on $\Sigma_{U_{P_1}}$.

Let $r = n-1$ or $r = n$. By the symmetry of the Dynkin diagram for D_n it suffices to consider the case $r = n$. We have

$$\Sigma_{U_{P_n}} = \left\{e_i + e_j \,\middle|\, 1 \leq i < j \leq n\right\}.$$

It is obvious that α_n is non-obtuse. The Weyl group W_{0,M_n} is isomorphic to $\mathfrak{S}_n$ acting on $e_1, \ldots, e_n$. It clearly acts transitively on $\Sigma_{U_{P_n}}$.

Let $1 < r < n-1$. Then both $e_r - e_{r+1}$ and $e_{r-1} + e_{r+1}$ lie in $\Sigma_{U_{P_r}}$ and satisfy $(e_r - e_{r+1}, e_{r-1} + e_{r+1}) = -1$. Hence, α_r is not non-obtuse. The highest root is $\alpha_0 = e_1 + e_2 = \alpha_1 + 2\sum_{k=j}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n$. Then α_0 and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\alpha_0) = 2 \neq 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof α_0 does not lie in the W_{0,M_r}-orbit of α_r.

(E) Inside $V = \{(x_1, \dots, x_8) \in \mathbb{R}^8 \mid x_6 = x_7 = -x_8\}$ we consider the root system of type E_6

$$\begin{aligned}\Sigma &= \{\pm e_i \pm e_j \mid 1 \le i < j \le 5\} \\ &\cup \left\{ \pm\frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i\right) \middle| \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{5} \nu(i) = 0 \right\}.\end{aligned}$$

A basis is given by $\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + \cdots + e_7)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$ for $3 \le i \le 6$.

Let $r = 1$ or $r = 6$. By the symmetry of the Dynkin diagram for E_6 it suffices to consider the case $r = 1$. We have

$$\Sigma_{U_{P_1}} = \left\{ \frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i\right) \middle| \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{5} \nu(i) = 0 \right\}.$$

To ease the notation we write $\alpha_\nu := \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i)$ for each $\nu \in (\mathbb{Z}/2\mathbb{Z})^5$. Let $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^5$ with $\sum_{i=1}^{5} \nu(i) = \sum_{i=1}^{5} \mu(i) = 0$. We observe that the cardinality of the set $\{i \mid \nu(i) \neq \mu(i)\}$ is even, hence equals 0, 2, or 4. Therefore, $|\{i \mid \nu(i) = \mu(i)\}|$ is either 1, 3, or 5. Thus, we compute

$$(\alpha_\nu, \alpha_\mu) = \frac{1}{4}\big(3 + |\{i \mid \nu(i) = \mu(i)\}| - |\{i \mid \nu(i) \neq \mu(i)\}|\big) \ge 0.$$

Therefore, α_1 is non-obtuse. The Weyl group W_{0,M_1} is the group $\mathfrak{S}_5 \ltimes \Gamma$ of type D_5 described in (D) (it acts on $e_1, \dots, e_5$, while leaving e_6, e_7 and e_8 fixed). Given $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^5$ with $\alpha_\nu, \alpha_\mu \in \Sigma_{U_{P_1}}$, we may view $\mu - \nu$ as an element of Γ, which maps α_ν to α_μ. Hence, W_{0,M_1} acts transitively on $\Sigma_{U_{P_1}}$.

Let $1 < r < 6$. The roots $\beta_1 = \sum_{k=1}^{5} \alpha_k = \frac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 - e_3 + e_4 - e_5)$ and $\beta_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 = e_5 + e_3$ both lie in $\Sigma_{U_{P_r}}$ and satisfy $(\beta_1, \beta_2) = -1$. Hence, α_r is not non-obtuse. Notice that $\beta_1 = w_r(\alpha_r)$, where $w_r \in W_{0,M_r}$ is given by

$$\begin{aligned} w_2 &= s_{\alpha_5} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4}, \\ w_3 &= s_{\alpha_1} s_{\alpha_5} s_{\alpha_2} s_{\alpha_4}, \\ w_4 &= s_{\alpha_5} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}, \\ w_5 &= s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}. \end{aligned}$$

Clearly, $w_r^{-1}(\beta_2) \in \Sigma_{U_{P_r}}$ and $(\alpha_r, w_r^{-1}(\beta_2)) = (\beta_1, \beta_2) = -1$.

The highest root is $\alpha_0 = \frac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 + e_3 + e_4 + e_5) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. Then α_0 and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\alpha_0) > 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof α_0 does not lie in the W_{0,M_r}-orbit of α_r.

(F) Inside $V = \{(x_1, \dots, x_8) \in \mathbb{R}^8 \mid x_7 = -x_8\}$ we consider the root system of type E_7

$$\begin{aligned}\Sigma &= \{\pm e_i \pm e_j \mid 1 \le i < j \le 6\} \cup \{\pm(e_8 - e_7)\} \\ &\cup \left\{ \pm\frac{1}{2}\left(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{\nu(i)} e_i\right) \middle| \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{6} \nu(i) \neq 0 \right\}.\end{aligned}$$

A basis is given by $\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + \cdots + e_7)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$ for $3 \leq i \leq 7$.

Let $r = 7$. We have

$$\begin{aligned}\Sigma_{U_{P_7}} &= \{e_6 \pm e_i \mid 1 \leq i \leq 5\} \cup \{e_8 - e_7\} \\ &\quad \cup \left\{ \frac{1}{2}\left(e_8 - e_7 + e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}e_i\right) \middle| \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{5}\nu(i) \neq 0 \right\}.\end{aligned}$$

(These are all the positive roots not lying in the root system of type E_6 considered in (E).) To ease the notation we write $\alpha_\nu := \frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{\nu(i)}e_i)$ for each $\nu \in (\mathbb{Z}/2\mathbb{Z})^6$. We compute $(e_6 \pm e_i, e_6 \pm e_j) = 1 \pm \delta_{i,j} \geq 0$, $(e_6 \pm e_i, e_8 - e_7) = 0$, $(e_6 \pm e_i, \alpha_\nu) = \frac{1}{2}(1 \pm (-1)^{\nu(i)}) \geq 0$, and $(e_8 - e_7, \alpha_\nu) = 1$ for all $1 \leq i, j \leq 5$ and $\nu \in (\mathbb{Z}/2\mathbb{Z})^6$ with $\alpha_\nu \in \Sigma_{U_{P_7}}$. Let $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^6$ with $\nu(6) = \mu(6) = 0$ and $\sum_{i=1}^{5}\nu(i) = \sum_{i=1}^{5}\mu(i) \neq 0$. We observe that the cardinality of the set $\{i \mid \nu(i) \neq \mu(i)\}$ is even, but not 6, hence equals 0, 2, or 4. Therefore, $|\{i \mid \nu(i) = \mu(i)\}|$ is either 2, 4, or 6. Thus, we compute

$$(\alpha_\nu, \alpha_\mu) = \frac{1}{4}\big(2 + |\{i \mid \nu(i) = \mu(i)\}| - |\{i \mid \nu(i) \neq \mu(i)\}|\big) \geq 0.$$

Therefore, α_7 is non-obtuse. The Weyl group W_{0,M_7} is the group generated by s_1 and the group $\mathfrak{S}_5 \ltimes \Gamma$ of type D_5 described in (D) (it acts on $e_1, \ldots, e_5$, while leaving e_6, e_7 and e_8 fixed). Given $\nu, \mu \in (\mathbb{Z}/2\mathbb{Z})^6$ with $\alpha_\nu, \alpha_\mu \in \Sigma_{U_{P_7}}$, we may view $\mu - \nu$ as an element of Γ (by forgetting the last entry), which maps α_ν to α_μ. Moreover, $\mathfrak{S}_5 \ltimes \Gamma$ clearly acts transitively on $\{e_6 \pm e_i \mid 1 \leq i \leq 5\}$. Together with

$$\begin{aligned} s_1(e_6 - e_1) &= e_6 - e_1 + \alpha_{(0,1,1,1,1,1)} = \alpha_{(1,1,1,1,1,0)}, \qquad \text{and} \\ s_1(\alpha_{(1,0,0,0,0,0)}) &= \alpha_{(1,0,0,0,0,0)} + \alpha_{(0,1,1,1,1,1)} = e_8 - e_7 \end{aligned}$$

it follows that $\Sigma_{U_{P_7}}$ is the W_{0,M_7}-orbit of α_7. Hence, W_{0,M_7} acts transitively on $\Sigma_{U_{P_7}}$.

Let $1 \leq r < 7$. The roots $\beta_1 = \sum_{k=1}^{6}\alpha_k = \alpha_{(0,0,1,1,0,1)}$ and $\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 = \alpha_{(1,1,0,0,1,0)}$ both lie in $\Sigma_{U_{P_r}}$ and satisfy $(\beta_1, \beta_2) = -1$. Hence, α_r is not non-obtuse. Notice that $\beta_1 = w_r(\alpha_r)$, where $w_r \in W_{0,M_r}$ is given by

$$\begin{aligned} w_1 &= s_{\alpha_6}s_{\alpha_5}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}, \\ w_2 &= s_{\alpha_6}s_{\alpha_5}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}, \\ w_3 &= s_{\alpha_6}s_{\alpha_1}s_{\alpha_5}s_{\alpha_2}s_{\alpha_4}, \\ w_4 &= s_{\alpha_6}s_{\alpha_5}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}, \\ w_5 &= s_{\alpha_6}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}, \\ w_6 &= s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}. \end{aligned}$$

Clearly, $w_r^{-1}(\beta_2) \in \Sigma_{U_{P_r}}$ and $(\alpha_r, w_r^{-1}(\beta_2)) = (\beta_1, \beta_2) = -1$.

The highest root is $\alpha_0 = e_8 - e_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Then α_0 and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\alpha_0) > 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof α_0 does not lie in the W_{0,M_r}-orbit of α_r.

(G) Inside $V = \mathbb{R}^8$ we consider the root system of type E_8

$$\Sigma = \left\{\pm e_i \pm e_j \,\middle|\, 1 \le i < j \le 8\right\}$$
$$\cup \left\{\pm\frac{1}{2}\left(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)} e_i\right) \,\middle|\, \nu(i) \in \mathbb{Z}/2\mathbb{Z}, \sum_{i=1}^{7}\nu(i) = 0\right\}.$$

A basis is given by $\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + \cdots + e_7)$, $\alpha_2 = e_2 + e_1$, and $\alpha_i = e_{i-1} - e_{i-2}$ for $3 \le i \le 8$.

Let $1 \le r \le 8$. The roots $\beta_1 = \sum_{k=1}^{8} \alpha_k = \frac{1}{2}(e_8 + e_7 - e_6 + e_1 + e_2 - e_3 - e_4 - e_5)$ and $\beta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 = \frac{1}{2}(e_8 + e_7 - e_6 - e_1 - e_2 + e_3 + e_4 + e_5)$ both lie $\Sigma_{U_{P_r}}$ and satisfy $(\beta_1, \beta_2) = -1$. Hence, α_r is not non-obtuse. Notice that $\beta_1 = w_r(\alpha_r)$, where $w_r \in W_{0,M_r}$ is given by

$$\begin{aligned}
w_1 &= s_{\alpha_8}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3},\\
w_2 &= s_{\alpha_8}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4},\\
w_3 &= s_{\alpha_8}s_{\alpha_7}s_{\alpha_6}s_{\alpha_1}s_{\alpha_5}s_{\alpha_2}s_{\alpha_4},\\
w_4 &= s_{\alpha_8}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2},\\
w_5 &= s_{\alpha_8}s_{\alpha_7}s_{\alpha_6}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4},\\
w_6 &= s_{\alpha_8}s_{\alpha_7}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5},\\
w_7 &= s_{\alpha_8}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6},\\
w_8 &= s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}.
\end{aligned}$$

Clearly, $w_r^{-1}(\beta_2) \in \Sigma_{U_{P_r}}$ and $(\alpha_r, w_r^{-1}(\beta_2)) = (\beta_1, \beta_2) = -1$.

The highest root is $\alpha_0 = e_8 + e_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_5 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$. Then α_0 and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\alpha_0) > 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof α_0 does not lie in the W_{0,M_r}-orbit of α_r.

(H) Inside $V = \mathbb{R}^4$ we consider the root system of type F_4

$$\Sigma = \{\pm e_i \mid 1 \le i \le 4\} \cup \left\{\pm e_i \pm e_j \,\middle|\, 1 \le i < j \le 4\right\} \cup \{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

A basis is given by $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$, and $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.

We have

$$\begin{aligned}
(\alpha_1, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4) &= (e_2 - e_3, e_1 + e_3) = -1,\\
(\alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4) &= (e_3 - e_4, e_1 - e_3) = -1,\\
(\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) &= (e_4, e_1 - e_4) = -1,\\
(\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4) &= \frac{1}{4}\cdot(e_1 - e_2 - e_3 - e_4, e_1 + e_2 + e_3 + e_4) = -\frac{1}{2}.
\end{aligned}$$

It follows that α_r is not non-obtuse for $1 \le r \le 4$.

Let $r = 1$ or $r = 2$. The highest root is $\alpha_0 = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Then both α_0 and α_r lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\alpha_0) >$

$1 = c_r(\alpha_r)$. By the remark at the beginning of the proof α_0 does not lie in the W_{0,M_r}-orbit of α_r.

Let $r = 3$ or $r = 4$. Consider the root $\beta = e_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. Then β and α_r both lie in $\Sigma_{U_{P_r}}$ and have the same length. But we have $c_r(\beta) > 1 = c_r(\alpha_r)$. By the remark at the beginning of the proof β does not lie in the W_{0,M_r}-orbit of α_r.

(I) Inside $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ consider the root system of type G_2

$$\Sigma = \pm\{e_1 - e_2, e_2 - e_3, e_1 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}.$$

A basis is given by $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_1 + e_2 + e_3$. Since we have

$$(\alpha_1, \alpha_1 + \alpha_2) = (e_1 - e_2, e_3 - e_1) = -1, \qquad \text{and}$$
$$(\alpha_2, 3\alpha_1 + \alpha_2) = (-2e_1 + e_2 + e_3, e_1 - 2e_2 + e_3) = -3,$$

neither α_1 nor α_2 are non-obtuse. The highest root is $\alpha_0 = 3\alpha_1+2\alpha_2 = -e_1-e_2+2e_3$. Then α_0 and α_2 both lie in $\Sigma_{U_{P_2}}$ and have the same length. But we have $c_2(\alpha_0) = 2 \neq 1 = c_2(\alpha_2)$. Hence α_0 does not lie in the W_{0,M_2}-orbit of α_2. Similarly, the roots $\beta = 2\alpha_1 + \alpha_2 = e_3 - e_2$ and α_1 both lie in $\Sigma_{U_{P_1}}$ and have the same length. As $c_1(\beta) = 2 \neq 1 = c_1(\alpha_1)$ it follows that β does not lie in the W_{0,M_1}-orbit of α_1.

The discussion above also proves the equivalences in (ii). □

As a consequence of Lemma 5.38 and Proposition 5.39 we make the following observation:

Corollary 5.40. *Let α_i be a non-obtuse simple root in $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Let $\alpha \in \Sigma_{U_{P_i}} = \Sigma^+ \setminus \Sigma_{M_i}$ with $\|\alpha\| = \|\alpha_i\|$. There exists a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ of the longest element w_0 in W_0 such that, if we put $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, there exists $0 \leq l < r$ with $\beta_1, \dots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = \alpha$. Moreover,*

$$\Sigma_{U_{P_i}} \setminus \{\alpha\} \subseteq \{\beta_{l+2}, \dots, \beta_r, -\beta_1, \dots, -\beta_l, -\alpha\} = \Sigma^+_{s_{i_1}\cdots s_{i_{l+1}}(\Delta)}.$$

Proof. Let w_{0,M_i} be the longest element in W_{0,M_i}. By [Bou81, Ch. VI, §1.6, Cor. 3 of Prop. 17] we have $w_0^2 = 1$ and $\ell(ww_0) = \ell(w_0) - \ell(w)$ for all $w \in W$, and similarly $w_{0,M_i}^2 = 1$ and $\ell(ww_{0,M_i}) = \ell(w_{0,M_i}) - \ell(w)$ for all $w \in W_{0,M_i}$.

Since α_i is non-obtuse, we find by Proposition 5.39 (ii) an element $w \in W_{0,M_i}$ with $w(\alpha_i) = \alpha$. Let $s_{i_1} \cdots s_{i_l}$ be a reduced decomposition of w and let $s_{i_{l+1}} \cdots s_{i_{r'}}$ be a reduced decomposition of $w_{0,M_i}w_0$. For every $v \in W_{0,M_i}$ we have

$$\ell(vw_{0,M_i}w_0) = \ell(w_0) - \ell(vw_{0,M_i}) = \ell(w_0) - \ell(w_{0,M_i}) + \ell(v) = \ell(v) + \ell(w_{0,M_i}w_0).$$

In particular, $s_{i_{l+1}} = s_i$, and $s_{i_1} \cdots s_{i_{r'}}$ is a reduced decomposition of $ww_{0,M_i}w_0$. If $s_{i_{r'+1}} \cdots s_{i_r}$ is a reduced decomposition of $v := w_0^{-1}w_{0,M_i}^{-1}w^{-1}w_0$, then $s_{i_1} \cdots s_{i_r}$ is a reduced decomposition of $w_0 = ww_{0,M_i}w_0v$. It is clear from the construction that $\beta_1, \dots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = \alpha$. The remaining assertion follows from Lemma 5.38. □

Notation 5.41. (a) Let $s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of the longest element w_0 of W_0. Define $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, for $1 \le j \le r$, as well as $\Delta^{(j)} := s_{i_1} \cdots s_{i_j}(\Delta)$ and

$$\Sigma^{(j)} := \Sigma^+_{\Delta^{(j)}} = \{\beta_{j+1}, \beta_{j+2}, \dots, \beta_r, -\beta_1, -\beta_2, \dots, -\beta_j\}$$

for $0 \le j \le r$; see Lemma 5.38.

For $k \ge 1$ let $1 \le j(k) \le r$ be the unique integer with $k \equiv j(k) \pmod r$. We put $\varepsilon_k := 1$ if $k \equiv j(k) \pmod{2r}$ and $\varepsilon_k := -1$ otherwise. Define $\beta_k := \varepsilon_k \beta_{j(k)}$, $\Delta^{(k)} := \varepsilon_k \Delta^{(j(k))}$, and

$$\Sigma^{(k)} := \varepsilon_k \Sigma^{(j(k))} = \Sigma^+_{\Delta^{(k)}} = \{\beta_{k+1}, \beta_{k+2}, \dots, \beta_{k+r}\}.$$

Keep in mind that the notation is always relative to a chosen reduced decomposition of w_0.

(b) For $\alpha \in \Phi$ we put $\varphi_{0,\varepsilon_\alpha \alpha} := \varepsilon_\alpha \varphi_{0,\alpha}$, where $\varepsilon_\alpha \in \mathbb{N}$ is the unique number with $\varepsilon_\alpha \alpha \in \Sigma$, see Remark 1.33, (a).

(c) For each $\alpha \in \Sigma$ we fix a lift $n_\alpha \in N \cap K$ of $s_\alpha \in W_0$ (cf. Remark 1.45).

(d) Given a basis Δ' of Σ, we denote by $U_{\Delta'}$ the group generated by $\bigcup_{\alpha \in \Sigma^+_{\Delta'}} U_\alpha$.

We recall that $K \cap U_\alpha = U_{(\alpha,0)}$ for all $\alpha \in \Sigma$.

Algorithm 5.42. Let $z, z' \in Z$ and $u \in U$ such that $uz' \in KzK$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of w_0. We put

$$u^{(0)} := \left(u^{(0)}_{\beta_r}, u^{(0)}_{\beta_{r-1}}, \dots, u^{(0)}_{\beta_1}\right) \quad \text{and} \quad z^{(0)} := z',$$

where the $u^{(0)}_\gamma \in U_\gamma$ are the unique elements such that $u^{(0)} = u$, by which we mean $u^{(0)}_{\beta_r} u^{(0)}_{\beta_{r-1}} \cdots u^{(0)}_{\beta_1} = u$.[11]

Suppose we have constructed $u^{(k)} = (u^{(k)}_{\beta_{k+r}}, u^{(k)}_{\beta_{k+r-1}}, \dots, u^{(k)}_{\beta_{k+1}}) \in U_{\Delta^{(k)}} \cap U_{\Delta^{(k-1)}}$ (we put $\Delta^{(-1)} := \Delta^{(0)}$) and $z^{(k)} \in Z$ with $u^{(k)} z^{(k)} \in KzK$. In order to define

$$u^{(k+1)} = \left(u^{(k+1)}_{\beta_{k+1+r}}, u^{(k+1)}_{\beta_{k+r}}, \dots, u^{(k+1)}_{\beta_{k+2}}\right) \in U_{\Delta^{(k+1)}} \cap U_{\Delta^{(k)}} \quad \text{and} \quad z^{(k+1)} \in Z$$

with $u^{(k+1)} z^{(k+1)} \in KzK$, there are three cases to consider.

(1) Case $\varphi_{0,\beta_{k+1}}(u^{(k)}_{\beta_{k+1}}) + \langle \beta_{k+1}, \nu(z^{(k)}) \rangle \ge 0$. This means $x := z^{(k),-1} u^{(k)}_{\beta_{k+1}} z^{(k)} \in U_{\beta_{k+1}} \cap K$. Define $u^{(k+1)} := u^{(k)} \cdot u^{(k),-1}_{\beta_{k+1}} \in U_{\Delta^{(k+1)}} \cap U_{\Delta^{(k)}}$, and $z^{(k+1)} := z^{(k)}$. We then have $u^{(k+1)} z^{(k+1)} = u^{(k)} z^{(k)} \cdot x^{-1} \in KzK$.

(2) Case $\varphi_{0,\beta_{k+1}}(u^{(k)}_{\beta_{k+1}}) \ge 0$ and not (1). Then $u^{(k)}_{\beta_{k+1}} \in K$. We define $u^{(k+1)} := u^{(k),-1}_{\beta_{k+1}} u^{(k)} \in U_{\Delta^{(k+1)}} \cap U_{\Delta^{(k)}}$, and $z^{(k+1)} := z^{(k)}$. In view of (DR2) and since $\Sigma^{(k)} \setminus \{\beta_{k+1}\} = \Sigma^{(k+1)} \cap \Sigma^{(k)}$ this is well-defined. We have $u^{(k+1)} z^{(k+1)} = u^{(k),-1}_{\beta_{k+1}} \cdot u^{(k)} z^{(k)} \in KzK$.

[11] We will identify the decomposition of any $u' \in U_{\Delta'}$ for some basis Δ' of Σ with u' itself.

(3) Case $f_k := \varphi_{0,\beta_{k+1}}(u^{(k)}_{\beta_{k+1}}) < \min\{0, -\langle \beta_{k+1}, \nu(z^{(k)})\rangle\}$. There exist unique $u'^{(k)}_{-\beta_{k+1}}, u''^{(k)}_{-\beta_{k+1}} \in U_{-\beta_{k+1}}$ with

$$m^{(k)} := u'^{(k)}_{-\beta_{k+1}} u^{(k)}_{\beta_{k+1}} u''^{(k)}_{-\beta_{k+1}} \in N,$$

cf. Remark 1.4, (b). By Proposition 1.27, (ii) the element $\nu(m^{(k)}) \in \operatorname{Aut}\mathcal{A}$ is the orthogonal reflection s_{β_{k+1},f_k}. Put $z^{(k+1)} := m^{(k)} z^{(k)} n_{\beta_{k+1}} \in N$. Since

$$\nu(z^{(k+1)}) = s_{\beta_{k+1},f_k}\big(\nu(z^{(k)})\big) = \nu(z^{(k)}) - \big(\langle \beta_{k+1}, \nu(z^{(k)})\rangle + f_k\big) \cdot \beta^\vee_{k+1} \tag{5.5.4}$$

is a translation on $\mathcal{A}$, we have $z^{(k+1)} \in Z$. Pairing β_{k+1} with (5.5.4) we obtain

$$\varphi_{0,\beta_{k+1}}\big(u^{(k)}_{\beta_{k+1}}\big) = f_k = -\frac{1}{2}\Big(\big\langle \beta_{k+1}, \nu(z^{(k)})\big\rangle + \big\langle \beta_{k+1}, \nu(z^{(k+1)})\big\rangle\Big). \tag{5.5.5}$$

By (V5) on page 23 we have $\varphi_{0,-\beta_{k+1}}\big(u'^{(k)}_{-\beta_{k+1}}\big) = -f_k > 0$ and $\varphi_{0,-\beta_{k+1}}\big(u''^{(k)}_{-\beta_{k+1}}\big) = -f_k > \langle \beta_{k+1}, \nu(z^{(k)})\rangle$, hence $u'^{(k)}_{-\beta_{k+1}}, z^{(k),-1} u''^{(k)}_{-\beta_{k+1}} z^{(k)} \in K$. We put

$$u^{(k+1)} := u'^{(k)}_{-\beta_{k+1}} \cdot u^{(k)} u^{(k),-1}_{\beta_{k+1}} \cdot u'^{(k),-1}_{-\beta_{k+1}} \in U_{\Delta^{(k+1)}} \cap U_{\Delta^{(k)}}.$$

We compute

$$\begin{aligned} u^{(k+1)} z^{(k+1)} &= u'^{(k)}_{-\beta_{k+1}} \cdot u^{(k)} u^{(k),-1}_{\beta_{k+1}} \cdot u'^{(k),-1}_{-\beta_{k+1}} \cdot m^{(k)} z^{(k)} n_{\beta_{k+1}} \\ &= u'^{(k)}_{-\beta_{k+1}} \cdot u^{(k)} z^{(k)} \cdot z^{(k),-1} u''^{(k)}_{-\beta_{k+1}} z^{(k)} n_{\beta_{k+1}} \in KzK. \end{aligned}$$

In this way we obtain a sequence $\big(u^{(k)}, z^{(k)}\big)_{k\geq 0}$ with $u^{(0)} = u$, $z^{(0)} = z'$, and $u^{(k)} z^{(k)} \in KzK$. Most importantly, the algorithm provides lower bounds for the values $\varphi_{0,\beta_{k+1}}(u^{(k)}_{\beta_{k+1}})$.

Proposition 5.43. *Let $z, z' \in Z$ and $u \in U$ with $uz' \in KzK$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of w_0. Then Algorithm 5.42 terminates, i. e. there is some $l \geq 0$ such that $u^{(l)} = 1$. Moreover, $\nu(z^{(l)})$ lies in the W_0-orbit of $\nu(z)$.*

Proof. As $\nu(Z)$ is a lattice in V its intersection with the closed ball of radius $\|\nu(z)\|$ around 0 is finite. This implies that there are only finitely many instances of case (3), since in case (3) $\nu(z^{(k+1)})$ is obtained by reflecting $\nu(z^{(k)})$ along the hyperplane $H_{(\beta_{k+1},f_k)}$. Since

$$\begin{aligned} &\langle \beta_{k+1}, 0\rangle + f_k = f_k < 0 \quad \text{and} \\ &\langle \beta_{k+1}, \nu(z^{(k)})\rangle + f_k < 0 \end{aligned}$$

it follows that 0 and $\nu(z^{(k)})$ lie on the same side of $H_{(\beta_{k+1},f_k)}$, whereas $\nu(z^{(k+1)})$ lies on the other. Now, elementary Euclidean geometry shows $\|\nu(z^{(k)})\| < \|\nu(z^{(k+1)})\|$.

Let $k \geq 0$ such that $\|\nu(z^{(k)})\|$ is maximal. Then $z^{(k+j)} = z^{(k)}$ for all $j \geq 0$. From the construction it is clear that $u^{(k+j)} \in U_{\Delta^{(k+j)}} \cap U_{\Delta^{(k)}}$ for $j \geq 0$, since from step k on only the cases (1) and (2) can occur. In particular, we have $u^{(k+r)} \in U_{\Delta^{(k+r)}} \cap U_{\Delta^{(k)}} = U_{-\Delta^{(k)}} \cap U_{\Delta^{(k)}} = \{1\}$. Moreover, we have $Kz^{(k+r)}K = KzK$, from which the last assertion follows. □

We now finish the proof of Theorem 5.35.

Proof of Theorem 5.35 (ii). Recall that we may assume that Φ is irreducible. Recall the special valuation φ_0 of $(Z, (M_\alpha, U_\alpha)_{\alpha\in\Phi})$ used to define $K = K_{\{\varphi_0\}}$. Let $\alpha_1, \dots, \alpha_n$ be the simple roots of Σ. We put $\Sigma_{U_P} := \Sigma^+ \smallsetminus \Sigma_M$. We denote by w_0 (resp. $w_{0,M}$) the longest element in W_0 (resp. $W_{0,M}$).

We are given $a \in Z$ strictly M-positive as well as $u \in U_P$, $z \in Z^-$, and $z' \in Z$ satisfying $\nu(z) \leq \nu(a^{-1})$ and $uz' \in KzK$. We need to show $aua^{-1} \in K_P$. Write $u = \prod_{\alpha\in\Sigma_{U_P}} u_\alpha$ for some ordering of the factors. Since $K_P \cap U_P = \prod_{\alpha\in\Sigma_{U_P}} U_{(\alpha,0)}$ and $U_{\beta,0} \cap U_{2\beta} = U_{2\beta,0}$, whenever $\beta, 2\beta \in \Phi_{U_P}$, it suffices to prove $\varphi_{0,\alpha}(au_\alpha a^{-1}) = \varphi_{0,\alpha}(u_\alpha) - \langle\alpha, \nu(a)\rangle \geq 0$, i. e.

$$\varphi_{0,\alpha}(u_\alpha) \geq \langle\alpha, \nu(a)\rangle \quad \text{for all } \alpha \in \Sigma_{U_P}. \tag{5.5.6}$$

Let α_0 be the highest root of Σ and write $\alpha_0 = \sum_{i=1}^n c_i(\alpha_0)\alpha_i$.

(a) Let α_{i_0} be the unique simple root in Σ_{U_P}. Let $\alpha \in \Sigma_{U_P}$ have the same length as α_{i_0}. By Corollary 5.40 we find a reduced decomposition $s_{i_1} \cdots s_{i_r}$ of w_0 such that for some $0 \leq l < r$ we have $\beta_1, \dots, \beta_l \in \Sigma_M$ and $\beta_{l+1} = \alpha$. We apply Algorithm 5.42. Since $\Sigma_{U_P} \smallsetminus \{\alpha\} \subseteq \Sigma^{(l+1)}$ it follows from (DR_2) that $u^{(l)}_{\beta_{l+1}} = u_\alpha$.

Notice that $-\langle\beta, \nu(z^{(k)})\rangle \geq \langle\beta, \nu(a)\rangle$ for all $\beta \in \Sigma_{U_P}$ and all $k \geq 0$ by Lemma 5.37. If $\varphi_{0,\alpha}(u_\alpha) \geq -\langle\beta_{l+1}, \nu(z^{(l)})\rangle$ or $\varphi_{0,\alpha}(u_\alpha) \geq 0$, then clearly $\varphi_{0,\alpha}(u_\alpha) \geq \langle\alpha, \nu(a)\rangle$. If however $\varphi_{0,\alpha}(u_\alpha) < \min\{0, -\langle\beta_{l+1}, \nu(z^{(l)})\rangle\}$ then (5.5.5) implies

$$\varphi_{0,\alpha}(u_\alpha) = -\frac{1}{2}\left(\langle\beta_{l+1}, \nu(z^{(l)})\rangle + \langle\beta_{l+1}, \nu(z^{(l+1)})\rangle\right) \geq \langle\alpha, \nu(a)\rangle.$$

When Σ is simply-laced, i. e. of type ADE, then all roots have the same length, and hence (5.5.6) holds.

It remains to study the cases where Σ is of type B_n or C_n (recall (B) and (C) in the proof of Proposition 5.39), and $\alpha \in \Sigma_{U_P}$ does not have the same length as α_{i_0}.

(b) Assume that Σ is of type B_n and that $\mathbf{P}$ corresponds to $\alpha_n = e_n$. We then have

$$\Sigma_{U_P} = \{e_i \mid 1 \leq i \leq n\} \cup \left\{e_i + e_j \,\middle|\, 1 \leq i < j \leq n\right\}.$$

If we write $u = \prod_{\alpha\in\Sigma_{U_P}} u_\alpha$ for some ordering $o\colon \Sigma \to \{1, 2, \dots, |\Sigma|\}$ of the factors (Definition 1.7, (b)), then $\varphi_{0,e_i+e_j}(u_{e_i+e_j})$ only depends on the relative position of u_{e_i} and u_{e_j}, while $\varphi_{0,e_i}(u_{e_i})$ does not depend on o. We choose o in such a way that $\varphi_{0,e_i}(u_{e_i}) \leq \varphi_{0,e_j}(u_{e_j})$ implies $o(e_i) \geq o(e_j)$. In order to ease the notation we assume $o(e_i) > o(e_j)$ if and only if $i < j$.

Since we have $s_1, \dots, s_{n-1} \in W_{0,M}$, $\ell(ww_{0,M}w_0) = \ell(w) + \ell(w_{0,M}w_0)$ for all $w \in W_{0,M}$, and $s_n s_i = s_i s_n$ for all $1 \le i \le n-2$, we deduce that a reduced decomposition of $w_{0,M}w_0$ necessarily starts with $s_n s_{n-1} \cdots$. Fix i, j with $1 \le i < j \le n$. By choosing $w \in W_{0,M} \cong \mathfrak{S}_n$ such that $w(e_n) = e_i$ and $w(e_{n-1}) = e_j$ we find, as in the proof of Corollary 5.40, a reduced decomposition $s_{i_1} \cdots s_{i_r}$ of w_0 such that for some $0 \le l \le r-2$ we have $\beta_1, \dots, \beta_l \in \Sigma_M$, $\beta_{l+1} = e_i$, and $\beta_{l+2} = e_i + e_j$.

We apply Algorithm 5.42. We have $u^{(l)} = u^{(0)}$ as elements of U_P, as well as $u^{(l)}_{\beta_{l+1}} = u_{e_i}$ and $u^{(l)}_{\beta_{l+2}} = u_{e_i+e_j}$.

If we are in case (1), i. e. $\varphi_{0,e_i}(u_{e_i}) \ge -\langle e_i, \nu(z')\rangle$, then clearly $u^{(l+1)}_{\beta_{l+2}} = u_{e_i+e_j}$, and by the same argument as in (a) we deduce $\varphi_{0,e_i+e_j}(u_{e_i+e_j}) \ge \langle e_i + e_j, \nu(a)\rangle$.

If we are in case (2), i. e. $\varphi_{0,e_i}(u_{e_i}) \ge 0$, then we have $u_{e_i}, u_{e_j} \in K$ (because we assumed $\varphi_{0,e_j}(u_{e_j}) \ge \varphi_{0,e_i}(u_{e_i})$ at the beginning). By (V3) we have $u_{e_i}^{-1}u_{e_j} = u_{e_j}u_{e_i}^{-1}[u_{e_i}, u_{e_j}^{-1}]$ with $[u_{e_i}, u_{e_j}^{-1}] \in U_{(e_i+e_j,0)}$. But this means $u^{(l+1)}_{\beta_{l+2}} = [u_{e_i}, u_{e_j}^{-1}]u_{e_i+e_j}$, and therefore either $\varphi_{0,\beta_{l+2}}(u^{(l+1)}_{\beta_{l+2}}) = \varphi_{0,e_i+e_j}(u_{e_i+e_j})$ or else $\varphi_{0,e_i+e_j}(u_{e_i+e_j}) \ge 0$. In either case we deduce by the same argument as in (a) that $\varphi_{0,e_i+e_j}(u_{e_i+e_j}) \ge \langle e_i + e_j, \nu(a)\rangle$.

If we are in case (3), i. e. $\varphi_{0,e_i}(u_{e_i}) < \min\{0, -\langle e_i, \nu(z')\rangle\}$, then we have $u^{(l+1)}_{\beta_{l+2}} = u_{e_i+e_j}$, and we again deduce $\varphi_{0,e_i+e_j}(u_{e_i+e_j}) \ge \langle e_i + e_j, \nu(a)\rangle$. This proves (5.5.6) (for the chosen ordering o of the factors).

(c) If Σ is of type C_n and $\mathbf{P}$ corresponds to $\alpha_n = 2e_n$, then by a similar argument as in (b) we deduce (5.5.6). The argument becomes easier, though, since U_P is commutative (by (DR2) together with $c_n(\alpha_0) = 1$).

(d) Assume that Φ is of type BC_n (hence Σ is of type B_n) and that $\mathbf{P}$ corresponds to $\alpha_1 = e_1 - e_2$. The other cases, where Φ is of type B_n or C_n, are proved in essentially the same way. We have

$$\Sigma_{U_P} = \{e_1\} \cup \{e_1 \pm e_i \mid 2 \le i \le n\}\,.$$

The numbers $\varphi_{0,e_1\pm e_i}(u_{e_1\pm e_i})$ do not depend on the ordering of the factors and were estimated in (a). If we have $2\varphi_{0,e_1}(u_{e_1}) \ge \varphi_{0,e_1-e_i}(u_{e_1-e_i}) + \varphi_{0,e_1+e_i}(u_{e_1+e_i})$ for some $2 \le i \le n$ and some ordering of the factors, we easily obtain $\varphi_{0,e_1}(u_{e_1}) \ge \langle e_1, \nu(a)\rangle$. Therefore, we assume from now on

$$2\varphi_{0,e_1}(u_{e_1}) < \varphi_{0,e_1-e_i}(u_{e_1-e_i}) + \varphi_{0,e_1+e_i}(u_{e_1+e_i}) \tag{5.5.7}$$

for all $2 \le i \le n$, and all orderings of the factors.

Claim 1. *The decomposition $s_{e_1} = (s_1 s_2 \cdots s_{n-1})s_n(s_{n-1}s_{n-2}\cdots s_1)$ is reduced.*

Proof of the claim. We write this decomposition as $s_{i_1} \cdots s_{i_{2n-1}}$ and, moreover, put $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for $1 \le j \le 2n-1$. We have

$$\beta_j = \begin{cases} s_1 \cdots s_{j-1}(e_j - e_{j+1}) = e_1 - e_{j+1}, & \text{for } 1 \le j \le n-1; \\ s_1 \cdots s_{n-1}(e_n) = e_1, & \text{for } j = n, \end{cases}$$

while for $1 \le j \le n-1$ we have

$$\begin{aligned}\beta_{2n-j} &= s_1 s_2 \cdots s_j s_{j+1} \cdots s_n s_{n-1} \cdots s_{j+1}(e_j - e_{j+1}) \\ &= s_1 \cdots s_j s_{e_{j+1}}(e_j - e_{j+1}) = s_1 \cdots s_j(e_j + e_{j+1}) \\ &= e_1 + e_{j+1}.\end{aligned}$$

Hence, the set $\mathfrak{T}_{s_{e_1}} = \{s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_{2n-1}}\}$ has cardinality $2n-1$. Proposition 1.36, (iv) implies $\ell(s_{e_1}) = |\mathfrak{T}_{s_{e_1}}| = 2n-1$. The claim follows. □

We extend $s_1 s_2 \cdots s_n s_{n-1} \cdots s_1$ to a reduced decomposition $s_{i_1} \cdots s_{i_r}$ of w_0 and apply Algorithm 5.42.

It will be necessary to narrow down the *support* of $u^{(k)}$, that is to say the set $\{\gamma \in \Sigma \mid u_\gamma^{(k)} \neq 1\}$. For $0 \le k \le n-2$ we consider the set

$$\begin{aligned}\Psi^{(k)} := &\left\{e_1 \pm e_i \,\middle|\, k+2 \le i \le n\right\} \cup \left\{e_i \,\middle|\, 1 \le i \le k+1\right\} \\ &\cup \left\{e_i \pm e_j \,\middle|\, 2 \le i \le k+1, i < j \le n\right\} \cup \left\{e_i \pm e_1 \,\middle|\, 2 \le i \le k+1\right\}.\end{aligned}$$

We have $\Psi^{(0)} = \Sigma_{U_P}$ and

$$\Psi^{(k)} = \left(\Psi^{(k-1)} \smallsetminus \{e_1 - e_{k+1}\}\right) \cup \{e_{k+1}, e_{k+1} - e_1\} \cup \left\{e_{k+1} \pm e_i \,\middle|\, k+2 \le i \le n\right\}$$

for $1 \le k \le n-2$. Notice that $\Psi^{(k)}$ is the closed subset of $\Sigma^{(k)}$ generated by $\Psi^{(k-1)} \smallsetminus \{e_1 - e_{k+1}\}$ and $e_{k+1} - e_1$. Under the addition map $\Psi^{(k)} \times \Psi^{(k)} \longrightarrow V^*$ the preimage of $\{e_1, 2e_1\}$ is the set of pairs $(e_1 \pm e_i, e_1 \mp e_i)$ for $k+2 \le i \le n$, while the preimage of $e_1 \pm e_i$ is empty for $k+2 \le i \le n$.

We now prove by induction on k that the support of $u^{(k)}$ is contained in $\Psi^{(k)}$. This is trivial for $k = 0$. Assume that $u^{(k)}$ is supported in $\Psi^{(k)}$ for some $k \ge 0$. In the cases (1) and (2) it follows that $u^{(k+1)}$ is supported in $\Psi^{(k)} \smallsetminus \{e_1 - e_{k+2}\}$. In case (3) the support of $u^{(k+1)}$ is contained in the closed subset of Σ generated by $\Psi^{(k)} \smallsetminus \{e_1 - e_{k+2}\}$ and $e_{k+2} - e_1$. In any case, $u^{(k+1)}$ is supported in $\Psi^{(k+1)}$. The induction is finished.

Claim 2. *We have* $\varphi_{0,e_1}(u_{e_1}^{(k)}) = \varphi_{0,e_1}(u_{e_1}^{(k-1)})$ *and* $u_{e_1 \pm e_i}^{(k)} = u_{e_1 \pm e_i}^{(k-1)}$ *for all* $0 \le k \le n-1$ *and all* $k+2 \le i \le n$. *(We put* $u_\gamma^{(-1)} := u_\gamma$ *for* $\gamma \in \Sigma$.*)*

Proof of the claim. We prove the claim by induction on k. For $k = 0$ there is nothing to show. Assume the claim holds for all $0 \le j \le k$ for some $0 \le k \le n-2$. If we are in case (1), i. e. $\varphi_{0,e_1-e_{k+2}}(u_{e_1-e_{k+2}}^{(k)}) \ge -\langle \beta_{k+1}, \nu(z^{(k)}) \rangle$, then clearly $u_\gamma^{(k+1)} = u_\gamma^{(k)}$ for all $\gamma \in \Psi^{(k)} \smallsetminus \{e_1 - e_{k+2}\}$. If we are in case (2), i. e. $\varphi_{0,\beta_{k+1}}(u_{\beta_{k+1}}^{(k)}) \ge 0$, then we have $u_{e_1 \pm e_i}^{(k+1)} = u_{e_1 \pm e_i}^{(k)}$ for all $k+3 \le i \le n$ by the above discussion of $\Psi^{(k)}$. Moreover, we have $u_{e_1}^{(k+1)} = u_{e_1}^{(k)} \cdot [u_{e_1+e_{k+2}}^{(k),-1}, u_{e_1-e_{k+2}}^{(k),-1}] \in U_{e_1}$. But the induction assumption implies $\varphi_{0,e_1}(u_{e_1}^{(k)}) = \varphi_{0,e_1}(u_{e_1})$ and also $u_{e_1 \pm e_{k+2}}^{(k)} = u_{e_1 \pm e_{k+2}}$. Using (V3) and (5.5.7) we compute

$$\begin{aligned}\varphi_{0,2e_1}\left([u_{e_1+e_{k+2}}^{(k),-1}, u_{e_1-e_{k+2}}^{(k),-1}]\right) &\ge \varphi_{0,e_1-e_{k+2}}(u_{e_1-e_{k+2}}^{(k)}) + \varphi_{0,e_1+e_{k+2}}(u_{e_1+e_{k+2}}^{(k)}) \\ &> 2\varphi_{0,e_1}(u_{e_1}^{(k)}).\end{aligned}$$

Hence also $\varphi_{0,e_1}\big([u^{(k),-1}_{e_1+e_{k+2}}, u^{(k),-1}_{e_1-e_{k+2}}]\big) > \varphi_{0,e_1}(u^{(k)}_{e_1})$ from which we conclude $\varphi_{0,e_1}(u^{(k+1)}_{e_1}) = \varphi_{0,e_1}(u^{(k)}_{e_1})$. Finally, if we are in case (3), we have $u^{(k+1)}_{e_1\pm e_i} = u^{(k)}_{e_1\pm e_i}$ for $k+3 \le i \le n$ and $u^{(k+1)}_{e_1} = u^{(k)}_{e_1}$ as follows again from the discussion of $\Psi^{(k)}$. The induction is finished. □

From Claim 2 we obtain $\varphi_{0,e_1}(u_{e_1}) = \varphi_{0,\beta_n}(u^{(n-1)}_{\beta_n}) \ge \langle e_1, \nu(a)\rangle$. Whence (5.5.6).

□

Remark 5.44. Recall that our strategy to prove the statement in Conjecture 5.23 was based on showing $(a_P)^i_{K_P} X_i \in C^+_P$ for all i. This approach is successful when **P** is non-obtuse. If, however, **P** is not non-obtuse, then we have $(a_P)_{K_P} X_1 \notin C^+_P$, and hence the strategy fails. To see this, let $\alpha, \beta \in \Sigma_{U_P}$ such that $\langle \alpha, \beta^\vee\rangle < 0$. Choose a lift $z \in Z$ of $\nu(s_\beta(-\lambda_P))$. The coefficient of $(z)_{K_P}$ in X_1 is non-zero. But since

$$\langle \alpha, \nu(a_P z)\rangle = \langle \alpha, \nu(\lambda_P) - s_\beta(\nu(\lambda_P))\rangle = \langle \beta, \nu(\lambda_P)\rangle \cdot \langle \alpha, \beta^\vee\rangle > 0,$$

we have $a_P z \notin M^+$. It follows from Proposition 4.9 that $(a_P)_{K_P} \cdot (z)_{K_P} \notin C^+_P$. But then we must also have $(a_P)_{K_P} X_1 \notin C^+_P$.

A. Appendix: The Hecke DGA

We recall the construction of the Hecke DGA as given in [Sch15]. We then present a candidate of a morphism of Hecke DGA's which is expected to be an analogon of the homomorphism Θ_M^P defined in Proposition 4.3.

A.1. Definition of the Hecke DGA

Definition A.1. (a) A *differential graded algebra* (short: *DGA*) is an algebra A together with a decomposition $A = \bigoplus_{n\in\mathbb{Z}} A^n$ of abelian groups, and an additive map $d\colon A \to A$ such that

(i) $A^n \cdot A^m \subseteq A^{n+m}$ for all $n, m \in \mathbb{Z}$;
(ii) $d(A^n) \subseteq A^{n+1}$ for all $n \in \mathbb{Z}$;
(iii) $d \circ d = 0$;
(iv) $d(ab) = d(a) \cdot b + (-1)^n a \cdot d(b)$ for all $a \in A^n$, $b \in A^m$, and $n, m \in \mathbb{Z}$.

(b) A *morphism of DGA's* $(A, d_A) \to (B, d_B)$ is an algebra map $A \to B$ satisfying $f(A^n) \subseteq B^n$, for all $n \in \mathbb{Z}$, and $f \circ d_A = d_B \circ f$.

Let G be a locally profinite group, and let k be a field. Recall that a G-representation on a k-vector space V is called *smooth* if the stabilizers $\{g \in G \,|\, gv = v\}$ are open subgroups of G for all $v \in V$. We denote by $\mathrm{Mod}_k(G)$ the category of smooth G-representations on k-vector spaces. Recall that $\mathrm{Mod}_k(G)$ has enough injectives [Vig96, I.5.9].

Fix a compact open subgroup $I \subseteq G$. Then

$$\mathrm{ind}_I^G(\mathbb{1}) := \{\text{finitely supported functions } I\backslash G \to k\}$$

is a smooth G-representation with G acting by right translations. Given $g \in G$, we denote by $[g] \in \mathrm{ind}_I^G(\mathbb{1})$ the function with $[g](Ig^{-1}) = 1$ and zero everywhere else. We have $g[h] = [gh]$ for all $g, h \in G$ and $f = \sum_{Ig\in I\backslash G} f(Ig) \cdot [g^{-1}]$ for all $f \in \mathrm{ind}_I^G(\mathbb{1})$.

We fix an injective resolution $\mathrm{ind}_I^G(\mathbb{1}) \to \mathcal{I}^\bullet$ in $\mathrm{Mod}_k(G)$.

Definition A.2. We call $\mathcal{H}_G^\bullet := \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$ the *Hecke DGA* attached to (I, G). More concretely, we have

$$\mathcal{H}_G^n = \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n}), \quad \text{for } n \in \mathbb{Z}.$$

Multiplication is given by

$$(ab)_q := (-1)^{mn} b_{q+n} \circ a_q \in \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n+m})$$

for all $a = (a_q)_q \in \mathcal{H}_G^n$, $b = (b_q)_q \in \mathcal{H}_G^m$, and $q \in \mathbb{Z}$. The differential is given by

$$(da)_q := d_{\mathcal{I}}^{q+n} \circ a_q - (-1)^n a_{q+1} \circ d_{\mathcal{I}}^q \in \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n+1})$$

for all $a \in \mathcal{H}_G^n$ and $q \in \mathbb{Z}$. (Here, $d_{\mathcal{I}}$ is the differential on $\mathcal{I}^\bullet$.)

Remark A.3. Up to quasi-isomorphism $\mathcal{H}_G^\bullet$ does not depend on the choice of the injective resolution, see the discussion following Remark 7 in [Sch15].

Lemma A.4. *The algebra $\mathcal{H}_G^\bullet$ is a DGA.*

Proof. It is clear from the definition that $\mathcal{H}_G^\bullet$ satisfies (i) and (ii) in Definition A.1. We verify (iii). Let $a \in \mathcal{H}_G^n$. For each $q \in \mathbb{Z}$ we compute

$$\begin{aligned}(d^2a)_q &= d_{\mathcal{I}}^{q+n+1} \circ (da)_q - (-1)^{n+1}(da)_{q+1} \circ d_{\mathcal{I}}^q \\ &= d_{\mathcal{I}}^{q+n+1} \circ d_{\mathcal{I}}^{q+n} \circ a_q - (-1)^n d_{\mathcal{I}}^{q+n+1} \circ a_{q+1} \circ d_{\mathcal{I}}^q \\ &\quad - (-1)^{n+1} d_{\mathcal{I}}^{q+1+n} \circ a_{q+1} \circ d_{\mathcal{I}}^q + (-1)^{n+1+n} a_{q+2} \circ d_{\mathcal{I}}^{q+1} \circ d_{\mathcal{I}}^q \\ &= 0,\end{aligned}$$

since $d_{\mathcal{I}} \circ d_{\mathcal{I}} = 0$. We verify (iv). Let $a \in \mathcal{H}_G^n$ and $b \in \mathcal{H}_G^m$. For each $q \in \mathbb{Z}$ we compute

$$\begin{aligned}(d(ab))_q &= d_{\mathcal{I}}^{q+n+m} \circ (ab)_q - (-1)^{n+m}(ab)_{q+1} \circ d_{\mathcal{I}}^q \\ &= (-1)^{nm} d_{\mathcal{I}}^{q+n+m} \circ b_{q+n} \circ a_q - (-1)^{n+m+nm} b_{q+1+n} \circ a_{q+1} \circ d_{\mathcal{I}}^q, \\ \big(d(a) \cdot b\big)_q &= (-1)^{m(n+1)} b_{q+n+1} \circ (da)_q \\ &= (-1)^{m(n+1)} b_{q+n+1} \circ d_{\mathcal{I}}^{q+n} \circ a_q - (-1)^{m(n+1)+n} b_{q+n+1} \circ a_{q+1} \circ d_{\mathcal{I}}^q, \\ \big((-1)^n a \cdot d(b)\big)_q &= (-1)^{n+n(m+1)} (db)_{q+n} \circ a_q \\ &= (-1)^{nm} d_{\mathcal{I}}^{q+n+m} \circ b_{q+n} \circ a_q - (-1)^{nm+m} b_{q+n+1} \circ d_{\mathcal{I}}^{q+n} \circ a_q.\end{aligned}$$

It follows that $d(ab) = d(a) \cdot b + (-1)^n a \cdot d(b)$. Hence, $\mathcal{H}_G^\bullet$ is a DGA. □

Remark A.5. We have (cf. [Har66, Cor. 6.5])

$$H^*(\mathcal{H}_G^\bullet) = \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}\big(\mathrm{ind}_I^G(\mathbb{1}), \mathrm{ind}_I^G(\mathbb{1})\big)^{\mathrm{op}}$$

as rings. In particular $H^0(\mathcal{H}_G^\bullet) = H_k(I, G)$.

A.2. A homomorphism between Hecke DGA's

Let F be a local field. Let $\mathbf{G}$ be a connected reductive group over F. Let $\mathbf{B}$ be a minimal parabolic subgroup of G. Let I be the pro-p-Iwahori subgroup associated with $\mathbf{B}$. Given a subset $X \subseteq G$, we write $I_X := I \cap X$. We fix a (standard) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi subgroup $\mathbf{M}$. The inclusion $M \subseteq P$ induces a restriction functor $\mathrm{Mod}_k(P) \to \mathrm{Mod}_k(M)$; implicitly we will view every smooth P-representation also as a smooth M-representation under this functor.

Consider the M-linear map

$$\vartheta\colon\ \mathrm{ind}_{I_P}^P(\mathbb{1}) \longrightarrow \mathrm{ind}_{I_M}^M(\mathbb{1}), \quad [g] \longmapsto [g_M],$$

where g_M is the image of $g \in P$ under the canonical projection map $P \twoheadrightarrow M$. Given an M-injective resolution $\mathrm{ind}_{I_M}^M(\mathbb{1}) \to \mathcal{I}^\bullet$, we wish to find a P-injective resolution $\mathrm{ind}_{I_P}^P(\mathbb{1}) \to \mathcal{J}^\bullet$ and a termwise surjective and M-linear morphism $\mathcal{J}^\bullet \to \mathcal{I}^\bullet$ of complexes.

Recall that the restriction functor $\mathrm{Mod}_k(P) \to \mathrm{Mod}_k(M)$ is left exact and has a right adjoint $\mathrm{Ind}_M^P\colon \mathrm{Mod}_k(M) \to \mathrm{Mod}_k(P)$ given by

$$\mathrm{Ind}_M^P(\mathcal{M}) = \left\{ f : P \to \mathcal{M} \;\middle|\; \begin{array}{l} f(mg) = mf(g) \text{ for all } m \in M,\, g \in P, \\ \text{and } \exists U \subseteq P \text{ open with } f(gu) = f(g) \\ \text{for all } g \in P,\, u \in U \end{array} \right\}$$

for each $\mathcal{M} \in \mathrm{Mod}_k(M)$. Again, P acts by right translation. The following lemma shows that Ind_M^P preserves injective objects.

Lemma A.6. *Let $R\colon \mathbf{C} \to \mathbf{D}$ be a functor with left adjoint $L\colon \mathbf{D} \to \mathbf{C}$. Assume that L preserves monomorphisms. Then R preserves injective objects.*

Proof. Confer [HS71, Prop. 10.2, p. 82]. Let $J \in \mathbf{C}$ be an injective object, and let $\iota\colon A \hookrightarrow B$ be a monomorphism in $\mathbf{D}$. Let $f : A \to R(J)$ be a morphism. We need to find a morphism $h\colon B \to R(J)$ such that $f = h \circ \iota$. As (L, R) is an adjoint pair we have a natural bijection

$$\alpha_{X,Y}\colon \mathrm{Hom}_{\mathbf{D}}(X, R(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(L(X), Y)$$

for each object $X \in \mathbf{D}$, $Y \in \mathbf{C}$. In particular, we have a morphism $\alpha_{A,J}(f)\colon L(A) \to J$. As L preserves monomorphisms, it follows that $L(\iota)\colon L(A) \hookrightarrow L(B)$ is a monomorphism. Since J is injective and $\alpha_{B,J}$ is surjective, there exists a morphism $h\colon B \to R(J)$ such that the morphism $\alpha_{B,J}(h)\colon L(B) \to J$ satisfies $\alpha_{B,J}(h) \circ L(\iota) = \alpha_{A,J}(f)$. Notice that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(B, R(J)) \arrow[r, "\iota^*"] \arrow[d, "\alpha_{B,J}"'] & \mathrm{Hom}_{\mathbf{D}}(A, R(J)) \arrow[d, "\alpha_{A,J}"] \\ \mathrm{Hom}_{\mathbf{C}}(L(B), J) \arrow[r, "L(\iota)^*"'] & \mathrm{Hom}_{\mathbf{C}}(L(A), J). \end{tikzcd}$$

Hence, we deduce $\alpha_{A,J}(f) = \alpha_{B,J}(h) \circ L(\iota) = \alpha_{A,J}(h \circ \iota)$. As $\alpha_{A,J}$ is injective we obtain $f = h \circ \iota$. Therefore, $R(J)$ is injective. □

Lemma A.7. *Let $\mathcal{M} \in \mathrm{Mod}_k(P)$, $\mathcal{N} \in \mathrm{Mod}_k(M)$, and $\varphi\colon \mathcal{M} \to \mathcal{N}$ be a surjective M-linear map. Let $\mathcal{N} \to \mathcal{I}^\bullet$ be an injective resolution in $\mathrm{Mod}_k(M)$.*

Then there exists an injective resolution $\mathcal{M} \to \mathcal{J}^\bullet$ in $\mathrm{Mod}_k(P)$ and a termwise M-linear and surjective morphism of complexes $\varphi^\bullet\colon \mathcal{J}^\bullet \to \mathcal{I}^\bullet$ extending φ.

Proof. Denote the injection $\mathcal{N} \to \mathcal{I}^0$ by $\varepsilon_{\mathcal{N}}$. Let $\mathcal{K}^0 \in \mathrm{Mod}_k(M)$ be an injective object with $\mathrm{Ker}(\varphi) \subseteq \mathcal{K}^0$. Put

$$\mathcal{J}^0 := \mathrm{Ind}_M^P(\mathcal{I}^0 \times \mathcal{K}^0) = \mathrm{Ind}_M^P(\mathcal{I}^0) \times \mathrm{Ind}_M^P(\mathcal{K}^0) \in \mathrm{Mod}_k(P).$$

Then $\mathcal{J}^0$ is injective by Lemma A.6. Since $\mathcal{K}^0$ is injective, we obtain an injective M-linear morphism $h\colon \mathcal{M} \to \mathcal{I}^0 \times \mathcal{K}^0$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\varphi"] \arrow[d, "h"'] & \mathcal{N} \arrow[d, "\varepsilon_{\mathcal{N}}"] \\ \mathcal{I}^0 \times \mathcal{K}^0 \arrow[r, "\mathrm{pr}_{\mathcal{I}^0}"'] & \mathcal{I}^0 \end{tikzcd}$$

commutes. Consider the P-linear map $\tau_{\mathcal{M}}\colon \mathcal{M} \to \operatorname{Ind}_M^P \mathcal{M}$ given by $\tau_{\mathcal{M}}(x)(g) = gx$. It splits the canonical M-linear surjection $\sigma_{\mathcal{M}}\colon \operatorname{Ind}_M^P \mathcal{M} \to \mathcal{M}$ given by $f \mapsto f(1)$. Since Ind_M^P is left exact we obtain an injective P-linear map $\varepsilon_{\mathcal{M}} = \operatorname{Ind}_M^P(h) \circ \tau_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{J}^0$. Let $\varphi^0\colon \mathcal{J}^0 \to \mathcal{I}^0$ be the map $\mathrm{pr}_{\mathcal{I}^0} \circ \sigma_{\mathcal{I}^0 \times \mathcal{K}^0}$. For each $x \in \mathcal{M}$ we compute

$$\sigma_{\mathcal{I}^0\times\mathcal{K}^0}\big(\varepsilon_{\mathcal{M}}(x)\big) = \sigma_{\mathcal{I}^0\times\mathcal{K}^0}\big((\operatorname{Ind}_M^P(h)\circ\tau_{\mathcal{M}})(x)\big) = h\big(\sigma_{\mathcal{M}}(\tau_{\mathcal{M}}(x))\big) = h(x).$$

Because of $\mathrm{pr}_{\mathcal{I}^0} \circ h = \varepsilon_{\mathcal{N}} \circ \varphi$, we obtain a commutative diagram (in $\mathrm{Mod}_k(M)$)

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\varphi"] \arrow[d, "\varepsilon_{\mathcal{M}}"'] & \mathcal{N} \arrow[r] \arrow[d, "\varepsilon_{\mathcal{N}}"] & 0 \\
\mathcal{J}^0 \arrow[r, "\varphi^0"] \arrow[d, two heads] & \mathcal{I}^0 \arrow[r] \arrow[d, two heads] & 0 \\
\operatorname{Coker}\varepsilon_{\mathcal{M}} \arrow[r] & \operatorname{Coker}\varepsilon_{\mathcal{N}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Notice that $\operatorname{Coker}\varepsilon_{\mathcal{M}}$ is a smooth P-representation. We now apply the above procedure with $(\mathcal{M}, \mathcal{N})$ replaced by $(\operatorname{Coker}\varepsilon_{\mathcal{M}}, \operatorname{Coker}\varepsilon_{\mathcal{N}})$ and thus find inductively an injective resolution $\mathcal{M} \to \mathcal{J}^\bullet$ in $\mathrm{Mod}_k(P)$ as desired. □

Remark A.8. In the situation of Lemma A.7 we find a P-linear, termwise splitting map $\psi^\bullet\colon \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ of the morphism $\varphi^\bullet$ constructed in the proof, where we consider $\mathcal{I}^n$, for $n \in \mathbb{Z}$, as a smooth P-module via inflation. It is termwise given by the composition $\mathcal{I}^n \xrightarrow{\tau_{\mathcal{I}^n}} \operatorname{Ind}_M^P \mathcal{I}^n \xrightarrow{\subseteq} \operatorname{Ind}_M^P \mathcal{I}^n \times \operatorname{Ind}_M^P \mathcal{K}^n = \mathcal{J}^n$.

The following lemma is useful:

Lemma A.9. *Let $\mathcal{N}$ be a smooth M-representation and consider the natural projection map $\sigma_{\mathcal{N}}\colon \operatorname{Ind}_M^P \mathcal{N} \twoheadrightarrow \mathcal{N}$, $f \mapsto f(1)$.*

(a) Let $U \subseteq \operatorname{Ker}(\sigma_{\mathcal{N}})$ be a P-invariant subspace. Then $U = \{0\}$.

(b) Let $\mathcal{M}$ be a second smooth M-representation, and let $F\colon \operatorname{Ind}_M^P \mathcal{M} \to \operatorname{Ind}_M^P \mathcal{N}$ be a P-linear map. Then there exists at most one M-linear map $F'\colon \mathcal{M} \to \mathcal{N}$ making the diagram

$$\begin{tikzcd}
\operatorname{Ind}_M^P \mathcal{M} \arrow[r, "F"] \arrow[d, "\sigma_{\mathcal{M}}"'] & \operatorname{Ind}_M^P \mathcal{N} \arrow[d, "\sigma_{\mathcal{N}}"] \\
\mathcal{M} \arrow[r, "F'"'] & \mathcal{N}
\end{tikzcd}$$

commutative. In this case we have $F' = \sigma_{\mathcal{N}} \circ F \circ \tau_{\mathcal{M}}$ and $F = \operatorname{Ind}_M^P(F')$.

Proof. (a) Let $f \in U$. Given $g \in P$, we have $f(g) = (g.f)(1) = \sigma_{\mathcal{N}}(gf) = 0$, since $g.f \in U \subseteq \operatorname{Ker}(\sigma_{\mathcal{N}})$. Therefore, $f = 0$.

(b) The uniqueness statement is clear since $\sigma_{\mathcal{M}}$ is surjective. Given F' such that the diagram commutes, it follows that $\operatorname{Im}(F - \operatorname{Ind}_M^P(F'))$ is a P-invariant subspace of

$\mathrm{Ker}(\sigma_{\mathcal{N}})$. From (a) we conclude $F = \mathrm{Ind}_M^P(F')$. Using the commutativity of the diagram and $\sigma_{\mathcal{M}} \circ \tau_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$, we see

$$\sigma_{\mathcal{N}} \circ F \circ \tau_{\mathcal{M}} = F' \circ \sigma_{\mathcal{M}} \circ \tau_{\mathcal{M}} = F'. \qquad \square$$

Fix an injective resolution $\mathrm{ind}_{I_M}^M(\mathbb{1}) \to \mathcal{I}^\bullet$ in $\mathrm{Mod}_k(M)$. By Lemma A.7 we find an injective resolution $\mathrm{ind}_{I_P}^P(\mathbb{1}) \to \mathcal{J}^\bullet$ together with a termwise surjective and M-linear map $\vartheta^\bullet\colon \mathcal{J}^\bullet \to \mathcal{I}^\bullet$, extending ϑ. Let $\mathcal{H}_P^\bullet$ and $\mathcal{H}_M^\bullet$ be the Hecke DGA's with respect to $\mathcal{J}^\bullet$ and $\mathcal{I}^\bullet$, respectively.

Given any four P-modules $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M}'$, and $\mathcal{N}'$, we make the natural identification

$$\mathrm{Hom}_P(\mathcal{M}\times\mathcal{N}, \mathcal{M}'\times\mathcal{N}') \xrightarrow{\sim} \begin{pmatrix} \mathrm{Hom}_P(\mathcal{M},\mathcal{M}') & \mathrm{Hom}_P(\mathcal{N},\mathcal{M}') \\ \mathrm{Hom}_P(\mathcal{M},\mathcal{N}') & \mathrm{Hom}_P(\mathcal{N},\mathcal{N}') \end{pmatrix},$$
$$f \longmapsto \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Proposition A.10. *(a) For $q, n \in \mathbb{Z}$ we put*

$$\mathrm{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+n}) := \left\{ f \in \mathrm{Hom}_P(\mathcal{J}^q, \mathcal{J}^{q+n}) \;\middle|\; \begin{array}{l} f_{11} \in \mathrm{Ind}_M^P(\mathrm{Hom}_M(\mathcal{I}^q, \mathcal{I}^{q+n})) \\ \textit{and } f_{12} = 0 \end{array} \right\},$$

and further $\mathrm{End}'^n_P(\mathcal{J}^\bullet) := \prod_{q\in\mathbb{Z}} \mathrm{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+n})$. *Then*

$$\mathcal{H}'^\bullet_P := \bigoplus_{n\in\mathbb{Z}} \mathrm{End}'^n_P(\mathcal{J}^\bullet)$$

is a sub-DGA of $\mathcal{H}_P^\bullet$.

(b) There exists a canonical surjective morphism $\rho^\bullet\colon \mathcal{H}'^\bullet_P \to \mathcal{H}_M^\bullet$ of DGA's.

Proof. (a) Since the matrices in $\mathrm{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+n})$ are lower triangular and Ind_M^P is a functor, it is easy to see that $\mathcal{H}'^\bullet_P$ is a graded subalgebra of $\mathcal{H}_P^\bullet$. Take $f \in \mathrm{Hom}_P(\mathcal{J}^q, \mathcal{J}^{q+n})$. Then we have $f \in \mathrm{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+n})$ if and only if there exists $\rho^n(f) \in \mathrm{Hom}_M(\mathcal{I}^q, \mathcal{I}^{q+n})$ making the diagram

$$\begin{array}{ccc} \mathcal{J}^q & \xrightarrow{f} & \mathcal{J}^{q+n} \\ {\scriptstyle \vartheta^q}\downarrow & & \downarrow{\scriptstyle \vartheta^{q+n}} \\ \mathcal{I}^q & \xrightarrow[\rho^n(f)]{} & \mathcal{I}^{q+n} \end{array} \tag{A.2.1}$$

commutative. (And in this case $\rho^n(f)$ is unique, since ϑ^q is surjective.) Indeed, similarly to Lemma A.9, (a) one proves that $\mathrm{Ind}_M^P \mathcal{K}^r$ is the maximal P-invariant subspace of $\mathrm{Ker}\,\vartheta^r$ for all $r \in \mathbb{Z}$. Therefore, if $\rho^n(f)$ as above exists, then the matrix representation of f must be lower triangular. Applying Lemma A.9, (b) to the induced diagram

$$\begin{array}{ccc} \mathrm{Ind}_M^P \mathcal{I}^q & \xrightarrow{f_{11}} & \mathrm{Ind}_M^P \mathcal{I}^{q+n} \\ {\scriptstyle \sigma_{\mathcal{I}^q}}\downarrow & & \downarrow{\scriptstyle \sigma_{\mathcal{I}^{q+n}}} \\ \mathcal{I}^q & \xrightarrow[\rho^n(f)]{} & \mathcal{I}^{q+n} \end{array}$$

now shows that $f_{11} \in \operatorname{Ind}_M^P \operatorname{Hom}_M(\mathcal{I}^q, \mathcal{I}^{q+n})$. Hence, $f \in \operatorname{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+n})$. The converse direction is clear.

The above discussion shows $d^q_{\mathcal{J}} \in \operatorname{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+1})$ for all $q \in \mathbb{Z}$. Consequently, $\mathcal{H}'^{,\bullet}_P$ is a sub-DGA of $\mathcal{H}^\bullet_P$.

(b) The above discussion shows that we have a unique map $\rho^\bullet : \mathcal{H}'^{,\bullet}_P \to \mathcal{H}^\bullet_M$ such that (A.2.1) commutes for all $n, q \in \mathbb{Z}$. It is straightforward to verify that $\rho^\bullet$ is even a morphism of DGA's.

We consider the map $s^\bullet : \mathcal{H}^\bullet_M \to \mathcal{H}'^{,\bullet}_P$ given by

$$(s^n(f))_q = \begin{pmatrix} \operatorname{Ind}_M^P(f_q) & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{Hom}'_P(\mathcal{J}^q, \mathcal{J}^{q+n})$$

for for all $q \in \mathbb{Z}$ and all $f \in \operatorname{End}^n_M(\mathcal{I}^\bullet)$. This defines a splitting of $\rho^\bullet$, whence the surjectivity of $\rho^\bullet$.

We remark that $s^\bullet$ is additive, degree preserving, and multiplicative. But neither does $s^\bullet$ preserve the unit nor the differential. □

Remark A.11. In view of Proposition A.10 there remain several open questions:

(a) Is the inclusion $\mathcal{H}'^{,\bullet}_P \subseteq \mathcal{H}^\bullet_P$ a quasi-isomorphism? If the answer is positive, then $\rho^\bullet$ would give back the map Θ^P_M on the 0-th cohomology.

(b) If the answer to the above question is negative, we can still ask whether the image of $H^0(\rho^\bullet)$ in $\mathcal{H}_R(M)$ contains the positive subalgebra $\mathcal{H}_R(M^+)$.

(c) There is also another approach: the inclusion $\operatorname{ind}^P_{I_P}(\mathbb{1}) \hookrightarrow \mathcal{J}^\bullet$, where we view $\operatorname{ind}^P_{I_P}(\mathbb{1})$ as a complex concentrated in degree 0, is a quasi-isomorphism. Therefore, the restriction map

$$\mathcal{H}^\bullet_P \longrightarrow \operatorname{Hom}^\bullet_P\bigl(\operatorname{ind}^P_{I_P}(\mathbb{1}), \mathcal{J}^\bullet\bigr) \cong (\mathcal{J}^\bullet)^{I_P}$$

is a quasi-isomorphism.[12] Similarly, we have a quasi-isomorphism

$$\mathcal{H}^\bullet_M \longrightarrow \operatorname{Hom}^\bullet_P\bigl(\operatorname{ind}^M_{I_M}(\mathbb{1}), \mathcal{I}^\bullet\bigr) \cong (\mathcal{I}^\bullet)^{I_M}.$$

Now, $\vartheta^\bullet$ induces by restriction a morphism of complexes

$$\vartheta^\bullet : (\mathcal{J}^\bullet)^{I_P} \longrightarrow (\mathcal{I}^\bullet)^{I_M}.$$

It is clear from the construction that $H^0(\vartheta^\bullet)$ gives back Θ^P_M. However, it is not clear whether the map on cohomology

$$H^*(\vartheta^\bullet) : H^*(\mathcal{H}^\bullet_P) \longrightarrow H^*(\mathcal{H}^\bullet_M)$$

is a morphism of graded algebras. What we can say, however, is that the diagram

$$\begin{tikzcd}
\mathcal{H}'^{,\bullet}_P \arrow[r, hook] \arrow[d, "\rho^\bullet"'] & \mathcal{H}^\bullet_P \arrow[r, "\mathrm{qis}"] & (\mathcal{J}^\bullet)^{I_P} \arrow[d, "\vartheta^\bullet"] \\
\mathcal{H}^\bullet_M \arrow[rr, "\mathrm{qis}"] & & (\mathcal{I}^\bullet)^{I_M}
\end{tikzcd}$$

is commutative.

[12] This is a formal consequence of [Har66, Lem. 6.2] in the framework of triangulated categories.

List of Symbols

$(a_P)^{-n}_{K_P}$	"negative power" of $(a_P)_{K_P}$; lies in $\mathcal{O}^+_P$, page 118
$(g)_\Gamma, (\Gamma g \Gamma)$	a basis element of the Hecke-ring $H_A(\Gamma, S)$, page 50
$(Z, (U_\alpha, M_\alpha)_{\alpha\in\Phi})$	root group datum of type Φ associated with G, page 17
$\mathfrak{C}$	fundamental alcove in $\mathscr{A}$, page 32
$\mathbf{C}$	connected center of $\mathbf{G}$, page 15
$\mathbf{G}$	a fixed connected reductive group over F, page 15
$\mathbf{H}$, H	algebraic groups are denoted by boldface letters, their F-rational points are denoted by the corresponding lightface letter, page 15
$\mathbf{H}^\circ$	connected component of $\mathbf{H}$, page 15
$\mathbf{M}_J$	$:= \mathbf{Z}_\mathbf{G}(\mathbf{T}_J)$, the (standard) Levi subgroup corresponding to $J \subseteq \Delta$, where $\mathbf{T}_J = \left(\bigcap_{\alpha\in\Phi_J} \operatorname{Ker}\alpha\right)^\circ$; is itself a connected reductive group, page 43
$\mathbf{N}_\mathbf{G}(\mathbf{T})$, $\mathbf{N}$	normalizer of $\mathbf{T}$ in $\mathbf{G}$, page 15
$\mathbf{P}_J$	(standard) parabolic subgroup corresponding to $J \subseteq \Delta$ with unipotent radical $\mathbf{U}_J$ and Levi component $\mathbf{M}_J$, page 43
$\mathbf{P}_J^{\mathrm{op}}$	opposite parabolic subgroup of $\mathbf{P}_J$, page 43
$\mathbf{T}$	a maximal F-split torus of $\mathbf{G}$, page 15
$\mathbf{U}_J$	unipotent radical of $\mathbf{P}_J$, page 43
$\mathbf{U}_J^{\mathrm{op}}$	unipotent radical of $\mathbf{P}_J^{\mathrm{op}}$, page 43
$\mathbf{Z}_\mathbf{G}(\mathbf{T})$, $\mathbf{Z}$	centralizer of $\mathbf{T}$ in $\mathbf{G}$, page 15
$\alpha^\vee$	coroot in V corresponding to $\alpha \in \Phi$ satisfying $\langle \alpha, \alpha^\vee \rangle = 2$, page 16
$\operatorname{Aut} \mathscr{A}$	automorphisms of $\mathscr{A}$; also identified with affine isomorphisms on V using φ_0 as the origin, page 31
$\chi_{a_P}(t)$	the unique polynomial in $1 + tH_R(K, G)[t]$ with $\chi^S_{a_P}(t) = \widetilde{\chi}_{a_P}(t)$, page 117
$\chi_{a_P}\big((a_P)_{K_P}\big)$	$:= \sum_{i=0}^{\lvert W_0^M \rvert} (a_P)^i_{K_P} \cdot X_i$, page 117
$\ell(w)$	length of w for $w \in W_0$ or $w \in W^{\mathrm{aff}}$ or $w \in W$ or $w \in W(1)$, page 35
$\ell_\beta(w)$	integer defined by $\ell_\beta(w) < \langle \beta, w(x) - \varphi_0 \rangle < \ell_\beta(w) + 1$, for $\beta \in \Sigma$, $w \in W$ and any $x \in \mathfrak{C}$, page 40

$\mathbf{FGrp}_1$	category with objects the pairs (Γ_0, Γ_1) of groups such that Γ_0 is normal in Γ_1, page 72
Γ'_α	$:= \{\varphi_{0,\alpha}(u) \mid u \in U^*_\alpha \text{ and } \varphi_{0,\alpha}(u) = \sup \varphi_{0,\alpha}(uU_{2\alpha})\} \subseteq \Gamma_\alpha$, page 30
Γ_α	$:= \varphi_\alpha(U^*_\alpha)$, the set of values under φ_α, $\alpha \in \Phi$, page 24
$\Gamma_{(g)}$	subgroup of Γ defined by $\Gamma \cap g^{-1}\Gamma g$, for g in a group containing Γ, page 48
$\mathcal{H}_R(G)$	the pro-p Iwahori-Hecke algebra of the reductive group G; defined as $H_R(I(1), G)$, page 52
$\mathcal{H}_R(M, G)$	the R-algebra $R \otimes_{\mathbb{Z}} \operatorname{Im} \Theta^P_{M,\mathbb{Z}}$, page 87
$\mathcal{H}_R(M^+)$	the positive subalgebra of $\mathcal{H}_R(M)$, page 58
$\mathfrak{m}_F$	maximal ideal of O_F, page 15
κ_F	residue field of F, page 15
κ_G	Kottwitz homomorphism $G \longrightarrow X^*\big(\mathbf{Z}(\widehat{\mathbf{G}})\big)^\sigma_{\mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{unr}})}$; $\mathbf{Z}(\widehat{\mathbf{G}})$ is the center of the Langlands dual of $\mathbf{G}$ and $\sigma \in \mathrm{Gal}(F^{\mathrm{unr}}/F)$ is the Frobenius automorphism, page 38
λ_P	image of a_P in Λ; it is strictly positive, page 117
Λ	$:= Z/Z_0$, a finitely generated abelian group with finite torsion; we have $W = \Lambda \rtimes W_0$, page 39
$\Lambda_{M^+_J}$	submonoid $\{\lambda \in \Lambda \mid \langle \alpha, \nu(\lambda) \rangle \leq 0 \quad \text{for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_J\}$ of Λ, page 46
$\mathfrak{g}$	Lie algebra of $\mathbf{G}$, page 15
$\mathscr{A}$	$:= \{\varphi_0 + v \mid v \in V\}$, the apartment of G with distinguished point φ_0; it is an affine space under V, page 30
$\mathscr{B}$	adjoint building of G given by $G \times \mathscr{A}/\sim$, page 37
$\mathcal{R}$	a system of representatives for κ_F in O_F, page 104
$\mathcal{R}_B$	a system of representatives for F/O_F in F, page 104
$\mathcal{S}_G, \mathcal{S}$	the Satake homomorphism $H_R(K, G) \to H_R(K_Z, Z)$, page 109
$\mathfrak{H}$	the set of hyperplanes in $\mathscr{A}$, page 30
$\mathfrak{H}(\mathfrak{C})$	set of walls of the alcove $\mathfrak{C}$, page 32
$\mathfrak{H}_w$	hyperplanes separating $\mathfrak{C}$ and $w\mathfrak{C}$, page 35
$\mathfrak{S}_n$	the group of permutations of $\{1, 2, \ldots, n\}$, page 68
$\mathfrak{T}_w$	reflections corresponding to $\mathfrak{H}_w$, page 35

$\mathscr{D}(\mathbf{H})$	derived subgroup of $\mathbf{H}$, page 15
$\mathscr{O}_P^+$	$:= C_P^+.H_R(K,G) \subseteq H_R(K_P,P)$, page 118
$\mathscr{O}_P^-$	$:= H_R(K,G).C_P^- \subseteq H_R(K_P,P)$, page 118
$\mu_\Gamma(g)$	integer given by $[\Gamma : \Gamma_{(g)}] = \lvert\Gamma\backslash\Gamma g\Gamma\rvert$, page 48
$\mu_{U_P}(w)$	a power of q given by $\mu_{U_P}(m)$, where $w \in W_M$ represents $I_M m I_M$, page 79
ν	group homomorphism $Z \to V$ given by $z.\varphi = \varphi + \nu(z)$ for $z \in Z$ and valuation φ, page 28
O_F	valuation ring of F, page 15
Ω	$:= \{nZ_0 \in W \mid \nu(n)(\mathfrak{C}) = \mathfrak{C}\}$, the set of elements of length zero in W, page 40
ω	normalized valuation $F \to \mathbb{Z} \cup \{\infty\}$ of F, page 15
$\omega \circ \chi$	the map $Z \to \mathbb{R}$ given by $z \mapsto \frac{1}{n} \cdot \omega((n\chi)(z))$ for $z \in Z$, and $\chi \in X^*(T)$, $n \in \mathbb{N}$ with $n\chi \in X^*(Z)$, page 29
$\langle \cdot, \cdot \rangle$	the W_0-invariant pairing $V^* \times V \to \mathbb{R}$ on V, page 16
Φ^{aff}	set $\left\{a_{\alpha,r} \,\middle\vert\, \alpha \in \Phi, r \in \Gamma'_\alpha\right\}$ of affine roots in $\mathscr{A}$, page 30
Φ, $\Phi(\mathbf{G},\mathbf{T})$	(relative) root system of $\mathbf{G}$ with respect to $\mathbf{T}$ inside V^*, page 15
$\Phi^\vee$	coroot system associated with Φ inside V, page 16
Φ_J	subroot system of Φ generated by a subset J of the basis Δ of Φ, page 43
Ψ_{red}	set of reduced roots in a subset Ψ of a root system, page 16
Σ	the unique reduced root system in V^* with affine Weyl group W^{aff}, page 33
$\Sigma^+_{\Delta'}$	positive roots of Σ with respect to the basis Δ', page 125
Σ^{aff}	$:= \Sigma \times \mathbb{Z}$, affine roots, page 33
Σ_{U_P}	$:= \Sigma^+ \smallsetminus \Sigma_M$ for a parabolic $\mathbf{P} = \mathbf{M}\mathbf{U}_P$, page 126
$\tau_w^{M,G,*}$	the basis element of $\mathcal{H}_R(M,G)$ given by $1 \otimes \mu_{U_P}(w) T_w^{M,*}$, page 95
$\tau_w^{M,G}$	the basis element of $\mathcal{H}_R(M,G)$ given by $1 \otimes \mu_{U_P}(w) T_w^{M}$, page 87
θ^+	the embedding $\mathcal{H}_R(M^+) \hookrightarrow \mathcal{H}_R(G)$, $T_m^M \mapsto T_m$, page 58
Θ_M^P	R-algebra homomorphism $H_R(\Gamma,P) \to H_R(\Gamma_M,M)$ given by $(g)_\Gamma \mapsto \nu_M(g)\mu_{U_P}(g)(g_M)_{\Gamma_M}$, page 72
$\theta_M^{L,G}$	R-algebra homomorphism $\mathcal{H}_R(M,G) \to \mathcal{H}_R(M,L)$ given by $\tau_w^{M,G} \mapsto \mu_{U_{P_L}}(w)\tau_w^{M,L}$, page 88

$\varepsilon_o(w,s)$	equals 1 if the gallery $(w\mathfrak{C}, ws\mathfrak{C})$ crosses $H_{wsw^{-1}}$ in o-positive direction, and -1 otherwise, page 42
$\varepsilon_{P,Q}$	the embedding $H_R(K_Q,Q) \hookrightarrow H_R(K_P,P)$ for parabolics $\mathbf{P} \subseteq \mathbf{Q} \subseteq \mathbf{G}$, page 111
φ_0	a fixed discrete special valuation on $(Z,(U_\alpha)_{\alpha\in\Phi}$ that is compatible with ω, page 29
φ	valuation $(\varphi_\alpha)_{\alpha\in\Phi}$ of a root group datum consisting of maps $\varphi_\alpha\colon U_\alpha \to \mathbb{R}\cup\{\infty\}$, page 23
${}^{\mathrm{v}}\nu$	canonical projection $N \to N/Z = W_0$, page 18
$\widetilde{\chi}_{a_P}(t)$	$:= \prod_{w\in W_0^M}(1 - w\star e^{-\lambda_P}\cdot t) \in 1 + tR[\Lambda][t]$, page 117
$\xi^G_{G,M}$	R-algebra homomorphism $\mathcal{H}_R(M,G)\to\mathcal{H}_R(G)$ given by $\tau_w^{M,G}\mapsto T_{n\lambda}^{-1}T_{e^{n\lambda}w}$ for some strictly positive $\lambda\in\Lambda$ and $n\gg 0$ with $e^{n\lambda}w \in W_{M^+}$, page 87
$\xi^G_{L,M}$	an embedding $\mathcal{H}_R(M,G)\to\mathcal{H}_R(L,G)$ satisfying $\theta_L^{L,G}\circ\xi^G_{L,M} = \xi^L_{L,M}\circ\theta_M^{L,G}$, page 88
ζ_M	the anti-automorphism of $\mathcal{H}_R(M)$ given by $T_m^M \mapsto T^M_{m^{-1}}$, page 96
ζ_P	the anti-automorphism of $H_R(I_P(1),P)$ given by $(g)_{I_P(1)} \mapsto (g^{-1})_{I_P(1)}$, page 96
a_P	a fixed strictly M-positive element for the parabolic $\mathbf{P} = \mathbf{MU}_P$, page 114
$a_{\alpha,r}$	the closed half-space $\{x \in \mathcal{A} \mid \langle\alpha, x-\varphi_0\rangle + r \geq 0\}$ of $\mathcal{A}$, page 30
$C(a)$	centralizer of $(a)_{I_P(1)}$ in $H_R\big(I_P(1),P\big)$ for a strictly positive element $a\in Z(M)$, page 68
C_P^+	centralizer of $(a_P)_{K_P}$ in $H_R(K_P,P)$, page 114
C_P^-	centralizer of $(a_P^{-1})_{K_P}$ in $H_R(K_P,P)$, page 116
$C_c^\infty(K\backslash G/K,R)$	R-algebra of compactly supported K-biinvariant functions $G\to R$, page 107
$c_{\tilde{s}}$	certain element of $R[Z_\kappa]$ for $\tilde{s} \in S^{\mathrm{aff}}(1)$, page 53
$E_o^{M,G}(w)$	$:= \mu_{U_P}(w)\cdot E_o(w)$ an alcove walk basis element of $\mathcal{H}_R(M,G)$, page 95
E_k	the $k\times k$ identity matrix, page 60
$E_o(w)$	element of $\mathcal{H}_R(G)$ corresponding to an alcove walk describing w depending on the orientation o, page 56
F	non-archimedean local field, page 15

$g = g_M g_U$	unique decomposition of $g \in P$ with $g_M \in M$ and $g_U \in U_P$, page 71
g_M	image of $g \in P$ in M under the canonical projection map, page 71
g_U	is by definition $g_M^{-1} g \in U_P$ for $g \in P$, page 71
H^g	$:= \{g^{-1}hg \mid h \in H\}$, the conjugate of a subgroup H by g^{-1}, page 71
h^g	$:= g^{-1}hg$, the conjugate of an element h by g^{-1}, page 71
$H_A(\Gamma, S)$	the Hecke-ring of the Hecke pair (Γ, S); it is given by $X_A(\Gamma, S)^\Gamma$, page 49
H_s	hyperplane fixed by $s \in S(\mathfrak{H})$, page 32
$H_{(\alpha,k)}$	equals $H_{\alpha,k}$; used when $\alpha \in \Sigma$ (compared to $\alpha \in \Phi$), page 34
$H_{\alpha,r}$	the hyperplane $\{x \in \mathscr{A} \mid \langle \alpha, x - \varphi_0 \rangle + r = 0\}$ of $\mathscr{A}$, page 30
$H_{o,+}$	positive half-space of the orientation o of $(\mathscr{A}, \mathfrak{H})$, page 42
I	the fixed Iwahori subgroup $K_{\mathfrak{C}}$ of G; it is contained in K, page 38
$I_M(1)$	$:= I(1) \cap M$, page 60
$I_P(1)$	$:= I(1) \cap P$, page 60
I_{U_J}	subgroup of $I(1)$ given by $I(1) \cap U_J = I \cap U_J$, page 45
K	the fixed maximal parahoric subgroup $K_{\{\varphi_0\}}$ of G, page 38
$K_{\mathfrak{F}}(1)$	pro-p radical of the parahoric subgroup $K_{\mathfrak{F}}$, page 38
$K_{\mathfrak{F}}$	$:= \operatorname{Ker} \kappa_G \cap P_{\mathfrak{F}}$, the parahoric subgroup of G corresponding to the face $\mathfrak{F}$, page 38
$m(u)$	unique element in M_α such that $u \in U_\alpha m(u) U_\alpha$, where $u \in U_{-\alpha}^*$, page 18
$M_J^{+,G}, M_J^+$	submonoid of M_J-positive elements of M_J (as a Levi subgroup in G); it contains K_J, page 45
$M_{\alpha,r}$	the set $M_\alpha \cap U_{-\alpha}\varphi_\alpha^{-1}(\{r\})U_{-\alpha} = m(U_{\alpha,r} \setminus U_{\alpha,r+})$, page 28
$P(\Sigma^\vee)$	coweight lattice of Σ, page 34
q	cardinality of the residue field κ_F, page 15
$q(\alpha, k)$	a power of q defined as $\lvert U_{(\alpha,k)}/U_{(\alpha,k+1)} \rvert$ for $(\alpha, k) \in \Sigma \times \mathbb{Z}$; it coincides with $q(H_{(\alpha,k)})$, page 78
$Q(\Sigma^\vee)$	coroot lattice of Σ, page 34
$q(H)$	integers defined such that $q(wH_s) = q_s$ for $w \in W^{\text{aff}}$, $s \in S^{\text{aff}}$, page 56

q_w	defined to be $q_{s_1} \cdots q_{s_{\ell(w)}}$, where $w = s_1 \cdots s_{\ell(w)} u$ is a reduced decomposition with $s_i \in S^{\mathrm{aff}}(1)$ and $u \in \Omega(1)$, page 54
$q_{\tilde{s}}, q_s$	a power of q given by $\lvert U_{\alpha_s,r_s}/U_{\alpha_s,r_s+}\rvert$ for $\tilde{s} \in S^{\mathrm{aff}}(1)$ lifting $s \in S^{\mathrm{aff}}$, page 52
$q_{v,w}$	unique positive integer satisfying $q_v q_w = q_{vw} q_{v,w}^2$, where $v, w \in W(1)$, page 55
$S(\mathfrak{H})$	set $\{s_H \mid H \in \mathfrak{H}\}$ of reflections in the hyperplanes of $\mathscr{A}$; in bijection with $\mathfrak{H}$, page 32
$S^{\mathrm{aff}}, S(\mathfrak{C})$	set of reflections in the walls of $\mathfrak{C}$, page 35
S_λ	the element $\sum_{w \in W_0/W_{0,\lambda}} w \star e^\lambda \in \mathbb{Z}[\Lambda]$ for $\lambda \in \Lambda_{Z^+}$, page 110
s_H	orthogonal reflection on $\mathscr{A}$ through $H \in \mathfrak{H}$, page 32
$s_{\alpha,r}$	the affine reflection on V given by $v \mapsto v - \big(\langle \alpha, v\rangle + r\big) \cdot \alpha^\vee$, page 31
$s_{\alpha,\alpha^\vee}, s_\alpha$	reflection on V^* given by $x \mapsto x - \langle x, \alpha^\vee\rangle \cdot \alpha$ for $\alpha \in \Phi$, page 16
T_s^*	the element $T_s - c_s$ of $\mathcal{H}_R(G)$ for $s \in S^{\mathrm{aff}}(1)$. It satisfies $T_s^* T_s = T_s T_s^* = q_s T_{s^2}$ and $T_{s^{-1}}^* T_s = T_s T_{s^{-1}}^* = q_s$, page 54
T_w, T_g	shorthand for $\big(I(1) g I(1)\big)$ where $I(1)gI(1)$ corresponds to $w \in W(1)$, page 52
T_w^*	equals $T_{s_1}^* \cdots T_{s_{\ell(w)}}^* T_u$ for a reduced decomposition $w = s_1 \cdots s_{\ell(w)} u$, where $s_i \in S^{\mathrm{aff}}(1)$ and $u \in \Omega(1)$; it satisfies $T_{w^{-1}}^* T_w = T_w T_{w^{-1}}^* = q_w$, page 54
U_α	root group of G corresponding to $\alpha \in \Phi$, page 17
$U_{(\alpha,k)}$	equals $U_{\beta,\varepsilon_\beta^{-1}k}$ for $\beta \in \Phi_{\mathrm{red}}$ with $\alpha = \varepsilon_\beta \beta \in \Sigma$, page 34
$U_{\alpha,r+}$	$:= \bigcup_{s>r} U_{\alpha,s}$, the elements of the root group U_α with valuation $> r$, page 24
$U_{\alpha,r,\kappa}$	$:= U_{\alpha,r}/U_{\alpha,r+}$, a finite subquotient of U_α of q-power order, page 24
U_{α_r}	$:= \varphi_\alpha^{-1}\big([r,\infty]\big)$, the elements of the root group U_α with valuation $\geq r$, page 23
U_α^*	$:= U_\alpha \smallsetminus \{1\}$, page 17
V	$:= \big(X_*(T)/X_*(C)\big) \otimes_{\mathbb{Z}} \mathbb{R}$, the $\mathbb{R}$-vector space for $\mathbf{G}$, page 16
W	$:= N/Z_0$, the Iwahori-Weyl group of G, page 39
$W(1)$	$:= N/Z_0(1)$, the pro-p Iwahori-Weyl group of G, page 39
W^{aff}	the affine Weyl group of G, page 32
W_0	$:= N/Z$, the finite Weyl group of $\mathbf{G}$, page 15

$W_{0,\lambda}$	stabilizer in W_0 of $\lambda \in \Lambda$, page 108
$W_{M_J^+}, W_{M_J^+}^G$	submonoid of positive elements in W_{M_J}; we have $W_{M_J^+} \cong I_{M_J} \backslash M_J^+ / I_{M_J}$ and $W_{M_J^+} = \Lambda_{M_J^+} \rtimes W_{0,M_J}$, page 46
$X^*(H)$	group of F-characters of $\mathbf{H}$, page 15
$X_*(H)$	group of F-cocharacters of $\mathbf{H}$, page 15
X_+	the element $(\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))_{K_B}$ in $H_R(K_B, B)$, page 106
X_-	the element $(\left(\begin{smallmatrix}\pi^{-1} & 0\\ 0 & 1\end{smallmatrix}\right))_{K_B}$ in $H_R(K_B, B)$, page 106
$X_A(\Gamma, S)$	the A-module with basis $\{(\Gamma g) \mid \Gamma g \in \Gamma \backslash S\}$ for a Hecke-pair (Γ, S), page 49
$Z(M)$	center of M, page 58
Z_0	$:= \operatorname{Ker} \kappa_Z$, the unique parahoric subgroup of Z with pro-p radical $Z_0(1)$, page 38
Z_κ	$:= Z_0/Z_0(1)$, a finite abelian group; fits into an exact sequence $1 \to Z_\kappa \to W(1) \to W \to 1$, page 39

References

[Abe16a] Noriyuki Abe. Modulo p parabolic induction of pro-p-Iwahori Hecke algebra. *Journal für die reine und angewandte Mathematik*, 2016. doi:10.1515/crelle-2016-0043.

[Abe16b] Noriyuki Abe. Parabolic inductions for pro-p-Iwahori Hecke algebras. *arXiv e-prints*, Dec 2016. arXiv:1612.01312.

[And77] Anatoli N. Andrianov. On factorization of Hecke polynomials for the symplectic groups of genus n. *Mathematics of the USSR-Sbornik*, 33(3):343–373, 1977. doi:10.1070/SM1977v033n03ABEH002428.

[AZ95] Anatoli N. Andrianov and Vladimir G. Zhuravlev. *Modular Forms and Hecke Operators*. American Mathematical Society, 1995.

[BK98] Colin J. Bushnell and Philip C. Kutzko. Smooth representations of reductive p-adic groups: structure theory via types. *Proceedings of the London Mathematical Society*, 77(03):582–634, 1998. doi:10.1112/S0024611598000574.

[BL94] Laure Barthel and Ron Livné. Irreducible modular representations of GL_2 of a local field. *Duke Mathematical Journal*, 75(2):261–292, 08 1994. doi:10.1215/S0012-7094-94-07508-X.

[Bor91] Armand Borel. *Linear Algebraic Groups*. Graduate texts in mathematics 126. Springer, 2nd edition, 1991.

[Bou81] Nicolas Bourbaki. *Groupes et algèbres de Lie: Chapitres 4, 5 et 6*. Éléments de Mathématiques. Masson, 1981.

[BT72] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local : I. données radicielles valuées. *Publications Mathématiques de l'IHÉS*, 41:5–251, 1972. URL: http://eudml.org/doc/103918.

[BT84] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Publications Mathématiques de l'IHÉS*, 60:5–184, 1984. URL: http://eudml.org/doc/104001.

[BZ76] Joseph Bernstein and Alexander Zelevinsky. Representations of the group $\mathrm{GL}(n, F)$ where F is a non-archimedean local field. *Russian Mathematical Surveys*, 31(3):1–68, jun 1976. doi:10.1070/rm1976v031n03abeh001532.

[Dąb94] Romuald Dąbrowski. Comparison of the Bruhat and the Iwahori Decompositions of a p-Adic Chevalley Group. *Journal of Algebra*, 167(3):704–723, 1994. doi:10.1006/jabr.1994.1208.

[Gar97] Paul B. Garrett. *Buildings and Classical Groups*. Springer, 1 edition, 1997. URL: http://www.math.umn.edu/~garrett/m/buildings/book.pdf.

[Gör07] Ulrich Görtz. Alcove walks and nearby cycles on affine flag manifolds. *Journal of Algebraic Combinatorics*, 26(4):415–430, Dec 2007. doi:10.1007/s10801-007-0063-6.

[Gri88] Valeri A. Gritsenko. Parabolic extensions of a Hecke ring of the general linear group. *Journal of Soviet Mathematics*, 43(4):2533–2540, Nov 1988. doi:10.1007/BF01374983.

[Gri90] Valeri A. Gritsenko. Expansion of Hecke polynomials of classical groups. *Mathematics of the USSR-Sbornik*, 65(2):333–356, 1990. doi:10.1070/SM1990v065n02ABEH001145.

[Gri92] Valeri A. Gritsenko. Parabolic extensions of the Hecke ring of the general linear group. II. *Journal of Soviet Mathematics*, 62(4):2869–2882, Dec 1992. doi:10.1007/BF01098922.

[Har66] Robin Hartshorne. *Residues and Duality*, volume 20 of *Lecture Notes in Mathematics*. Springer, 1966.

[Her11a] Florian Herzig. A Satake isomorphism in characteristic p. *Compositio Mathematica*, 147(1):263–283, 2011. doi:10.1112/S0010437X10004951.

[Her11b] Florian Herzig. The classification of irreducible admissible mod p representations of a p-adic GL_n. *Inventiones Mathematicae*, 186(2):373–434, Nov 2011. doi:10.1007/s00222-011-0321-z.

[HR09] Thomas J. Haines and Sean Rostami. The Satake isomorphism for special maximal parahoric Hecke algebras. *ArXiv e-prints*, jul 2009. arXiv:0907.4506.

[HS71] Peter J. Hilton and Urs Stammbach. *A Course in Homological Algebra*. Graduate Texts in Mathematics 4. Springer New York, 1 edition, 1971.

[HT01] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties. (AM-151)*. Princeton University Press, 2001. with an appendix by Vladimir G. Berkovich.

[Hum98] James E. Humphreys. *Linear Algebraic Groups*. Graduate texts in mathematics 021. Springer, 4 edition, 1998.

[HV15] Guy Henniart and Marie-France Vignéras. A Satake isomorphism for representations modulo p of reductive groups over local fields. *Journal für die reine und angewandte Mathematik*, 701:33–75, 2015. doi:10.1515/crelle-2013-0021.

[Jan14] Fabian Januszewski. On p-adic L-functions for $\mathrm{GL}(n)\times\mathrm{GL}(n-1)$ Over Totally Real Fields. *International Mathematics Research Notices*, 2015(17):7884–7949, 2014. doi:10.1093/imrn/rnu181.

[Kot97] Robert E. Kottwitz. Isocrystals with additional structure. II. *Compositio Mathematica*, 109:255–339, 1997. doi:10.1023/A:1000102604688.

[Lam99] Tsit-Yuen Lam. *Lectures on modules and rings*, volume 189. Springer-Verlag New York, Inc., 1999.

[Lan01] Joshua M. Lansky. Decomposition of double cosets in p-adic groups. *Pacific Journal of Mathematics*, 197(1):97–117, 2001. doi:10.2140/pjm.2001.197.97.

[Mac71] Ian G. Macdonald. *Spherical functions on a group of p-adic type*. Ramanujan Institute, University of Madras, Madras 5, India, nov 1971.

[Oll10] Rachel Ollivier. Parabolic induction and Hecke modules in characteristic p for p-adic GL_n. *Algebra Number Theory*, 4(6):701–742, 2010. `doi:10.2140/ant.2010.4.701`.

[Oll14] Rachel Ollivier. Compatibility between Satake and Bernstein isomorphisms in characteristic p. *Algebra Number Theory*, 8(5):1071–1111, 2014. `doi:10.2140/ant.2014.8.1071`.

[Oll15] Rachel Ollivier. An inverse Satake isomorphism in characteristic p. *Selecta Mathematica*, 21(3):727–761, Jul 2015. `doi:10.1007/s00029-014-0157-7`.

[OV18] Rachel Ollivier and Marie-France Vignéras. Parabolic induction in characteristic p. *Selecta Mathematica*, Sep 2018. `doi:10.1007/s00029-018-0440-0`.

[Ren10] David Renard. *Représentations des groupes réductifs p-adiques*, volume 17 of *Cours spécialisés*. Société Mathématiques de France, 2010. URL: `http://www.cmls.polytechnique.fr/perso/renard/Padic.pdf`.

[Sch09] Nicolas A. Schmidt. *Generische pro-p Hecke-Algebren*. Diplomarbeit, Humboldt-Universität zu Berlin, 2009. URL: `http://www2.mathematik.hu-berlin.de/~schmidtn/hecke.pdf`.

[Sch15] Peter Schneider. Smooth representations and Hecke modules in characteristic p. *Pacific Journal of Mathematics*, 279, 12 2015. `doi:10.2140/pjm.2015.279.447`.

[Sch19] Nicolas A. Schmidt. *Generic pro-p Hecke algebras, the Hecke algebra of* $\mathrm{PGL}(2, \mathbb{Z})$, *and the cohomology of root data*. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2019. `doi:10.18452/19724`.

[Vie12] Mathieu Vienney. *Construction de (φ, Γ)-modules en caractéristique p*. PhD thesis, Ecole normale supérieure de Lyon – ENS LYON, 2012.

[Vig] Marie-France Vignéras. The pro-p Iwahori Hecke algebra of a reductive p-adic group IV (Levi subgroup and central extension). In preparation.

[Vig89] Marie-France Vignéras. Représentations modulaires de $\mathrm{GL}(2, F)$ en caractéristique ℓ, F corps p-adique, $p \neq \ell$. *Compositio Mathematica*, 72(1):33–66, 1989. URL: `http://www.numdam.org/item/CM_1989__72_1_33_0`.

[Vig96] Marie-France Vignéras. Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$. *Progress in Mathematics*, 137, Birkhäuser: Boston, 1996.

[Vig98] Marie-France Vignéras. Induced R-representations of p-adic reductive groups. *Selecta Mathematica*, 4(4):549–623, 1998. `doi:10.1007/s000290050040`.

[Vig04] Marie-France Vignéras. Representations modulo p of the p-adic group $\mathrm{GL}(2, F)$. *Compositio Mathematica*, 140:333–358, 2004. `doi:10.1112/S0010437X03000071`.

[Vig05] Marie-France Vignéras. Pro-p-Iwahori Hecke algebra and supersingular $\overline{\mathbb{F}}_p$-representations. *Mathematische Annalen*, 331(3):523–556, Mar 2005. `doi:10.`

1007/s00208-004-0592-4.

[Vig06] Marie-France Vignéras. Algèbres de Hecke affines génériques. *Representation Theory*, 10, jan 2006. doi:10.1090/S1088-4165-06-00185-3.

[Vig15] Marie-France Vignéras. The pro-p Iwahori Hecke algebra of a reductive p-adic group V (parabolic induction). *Pacific Journal of Mathematics*, 279:499–529, 2015. doi:10.2140/pjm.2015.279.499.

[Vig16] Marie-France Vignéras. The pro-p Iwahori Hecke algebra of a reductive p-adic group I. *Compositio Mathematica*, 152(4):693–753, 2016. doi:10.1112/S0010437X15007666.

Erklärung

Ich erkläre, dass ich die Dissertation selbstständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfmittel angefertigt habe.

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